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CONCERNING THE SPANS OF CERTAIN PLANE SEPARATING CONTINUA

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ABSTRACT. Let X be a plane separating continuum. Suppose C is a convex space contained in a bounded component of $R^2 - X$. It is shown that the span of the boundary of C is a lower bound for both the span and semispan of X. It is also shown that if a span of X is equal to the breadth of X and Y satisfies certain conditions relative to X then that span of X is an upper bound for the corresponding span of Y.

1. INTRODUCTION

The concept of the span of a metric space was introduced in [L1]. Various modified versions of the span have been defined since then (cf [L2] and [L3]). Questions, about how these various spans are related for simple closed curves, have motivated work in this area. The following question by H. Cook, has also generated interest.

> If S_1 and S_2 are two simple closed curves in the plane and S_2 is contained in the bounded component of $R^2 - S_1$, then is the span of S_1 larger than the span of S_2? ([CIL,pg391]).

K. Tkaczyńska has obtained some partial answers to these questions (also see [W1], [W2], [W3]). She has shown that if C is a convex space in the plane then each of the spans of ∂C are equal to the breadth of ∂C ([T1]). Later in [T2], she showed that if X is a simple closed curve in the plane and C is a convex region in the bounded complement of X, then the span of X is larger than or equal to the span of ∂C. In this paper we extend this result to cover any X, which is a plane separating continuum (see Theorem 2). For another partial solution to Cook's problem see [W1].

The following result is also given. If X is a continuum, where a span, τ, of X is the breadth of X and either $Y \subseteq X$ or X separates R^2 and $Y \subseteq R^2 - U$, where U is the unbounded component of the complement of $R^2 - X$, then $\tau(Y) \leq \tau(X)$.

2. Preliminaries

If X is a non-empty metric space, we define the *span* $\sigma(X)$ of X to be the least upper bound of the set of real numbers α which satisfy the following condition: there exists a connected space C and continuous mappings $f_1,\ f_2 : C \to X$ such that

$$(\sigma) \qquad f_1(C) = f_2(C)$$

and $\alpha \leq dist[f_1(c), f_2(c)]$ for $c \in C$.

The definition does not require X to be connected, but to simplify our discussion we will now consider X to be connected. The surjective span $\sigma^*(X)$, the semispan $\sigma_0(X)$, and the surjective semispan $\sigma_0^*(X)$ are defined as above, except we change conditions (σ) to the following :

$$(\sigma^*) \qquad f_1(C) = f_2(C) = X,$$

$$(\sigma_0) \qquad f_1(C) \subseteq f_2(C),$$

$$(\sigma_0^*) \qquad f_1(C) \subseteq f_2(C) = X,$$

The following inequalities follow immediately from the definitions.

$$\begin{array}{ccccccccc} 0 & \leq & \sigma^*(X) & \leq & \sigma(X) & \leq & \sigma_0(X) & \leq & diam\, X, \\ 0 & \leq & \sigma^*(X) & \leq & \sigma_0^*(X) & \leq & \sigma_0(X) & \leq & diam\, X. \end{array}$$

The notion of a directional diameter of a simple closed curve was given in [T2]. We can extend the definition of directional diameter to all planar continua. Let X be a planar continuum. Let L_α denote the line passing through the origin such that the angle between the positive x-axis and L_α, measured counterclockwise, is α, where $\alpha \in [0, \pi)$. The directional diameter $d_\alpha(X)$ of X, in the direction α, is the length of the longest line segment (or segments) with endpoints on X, that is parallel to L_α. In [T2] the breadth of a continuum was defined to be the $\inf\{d_\alpha(X) : \alpha \in [0, \pi)\}$. The breadth of a continuum was original denoted by $d(X)$ in [T1]. In this paper we will denote the breadth of X by $b(X)$, so that the name and the notation will correspond.

In the proof of Theorem 2, we use the following theorem from [L2, section 7].

Theorem L. *If X is a closed subset of the Hilbert cube and $I^\omega f : X \longrightarrow S$ is an essential mapping of X into the circumference S, then*

$$\inf_{s\in S} \rho(f^{-1}(s),\, f^{-1}(-s)) \leq \sigma(X).$$

3. Main Results

Theorem 1. *Let X be a continuum in R^2 such that $\tau(X) = b(X)$, where $\tau = \sigma$, σ_0, σ^*, or σ_0^*. Suppose Y is a continuum. If either $Y \subseteq X$ or X separates R^2 and $Y \subseteq R^2 - U$, where U is the unbounded complement of $R^2 - X$, then $\tau(Y) \leq \tau(X)$.*

Proof. For each $\varepsilon > 0$, there exists $\alpha\epsilon[0, \pi)$ such that $d_\alpha(X) - b(X) \leq \varepsilon$. Let A_α be a line segment parallel to L_α, such that endpoints a and b of A_α are elements of X and $\rho(a, b) = d_\alpha(X)$.

We can rotate and translate X in R^2, such that a is moved to the origin and b is moved to the point $(d_\alpha(X), 0)$.

Let $f, g : Z \longrightarrow Y$ be continuous functions, Z connected, and condition (τ) holds where:

$$(\sigma) \qquad g(Z) = f(Z)$$

$$(\sigma_0) \qquad g(Z) \subseteq f(Z)$$

$$(\sigma^*) \qquad g(Z) = f(Z) = Y$$

$$(\sigma_0^*) \qquad g(Z) \subseteq f(Z) = Y.$$

Let $d = \max\{t \mid \text{the graph of } y = t \text{ intersects } f(Z)\}$ and let $c = \min\{t \mid \text{the graph of } y = t \text{ intersects } f(Z)\}$. There are points z' and z'' of Z such that $p_2 \circ f(z') = c$ and $p_2 \circ f(z'') = d$, where $p_2 : R^2 \longrightarrow R$ is the projection map given by $p_2(x, y) = y$. We observe that $c = p_2 \circ f(z') \leq p_2 \circ g(z')$ and $p_2 \circ g(z'') \leq p_2 \circ f(z'') = d$. Consequently, there exists $z^* \in Z$ such that $p_2 \circ f(z^*) = p_2 \circ g(z^*)$. So, either $f(z^*) = g(z^*)$ or $f(z^*)$ and $g(z^*)$ are endpoints of an arc, A, which is parallel to the x-axis. In the latter case, there is an arc B, with endpoints in X such that $A \subseteq B$. By the definition of $d_\alpha(X)$, we see that $\rho(f(z^*), g(z^*)) = l(A) \leq l(B) \leq d_\alpha(X)$. Consequently, in either case, we get $\tau(Y) \leq \rho(f(z^*), g(z^*)) \leq d_\alpha(X) \leq b(X) + \varepsilon$. Since this is true for any $\varepsilon > 0$, we get that $\tau(Y) \leq \tau(X)$. □ □

Corollary 1.1. *Let X be a planar simple closed curve such that the bounded component B, of $R^2 - X$ is convex. For any continuum $Y \subseteq X \cup B$, $\tau(Y) \leq \tau(X)$ where $\tau = \sigma$, σ_0, σ^*, or σ_0^*.*

Proof. This follows immediately, since by [Theorem 3 of T1], $\tau(X) = b(X)$. □

Corollary 1.2. *Let Q be a planar quadrilateral. For any continuum $Y \subseteq Q \cup B$, where B is the bounded component of $R^2 - Q$, $\tau(Y) \leq \tau(Q)$ for $\tau = \sigma$, σ_0, σ^*, or σ_0^*.*

Proof. This follows from the fact that $\tau(Q) = b(X)$, which was shown in [T1]. □

The corresponding result also holds for the class of polygons described in [T1, example 3].

In Theorem 2, we consider X to be either in the real plane or the complex plane. We use the one that simplifies the exposition.

Theorem 2. *Let X be a separating planar continuum and let C be a convex region contained in a bounded component of the complement of X. Then $\sigma(X) \geq \sigma(\partial C)$.*

Proof. Let D be the simple closed curve such that $D = \partial C$. Pick $< b(D) \leq diam\, D$. We inscribe a convex polygon P in D such that each vertex of P is an element of D and $H(P, D) < \frac{\varepsilon}{2}$, where H is the Hausdorff metric. Clearly, $b(D) \geq b(P) \geq b(D) - \varepsilon$. In order to simplify the exposition, we will assume that no three vertices are on the same straight line.

Let a, b be vertices of P such that $dist\,(a, b) = diam\, P$. We can rotate and translate our whole space so that b is moved to the origin and a is moved to the point $(0, diam\, P)$. Note that the x-axis intersects P only at the point b and the line through a parallel to the x-axis, L, intersects P only at a. This is true since a and b are of distance the diameter of P apart.

The vertices of P have a clockwise ordering. Let a' be the next vertex of P in clockwise ordering after a. Let b' be the next vertex of P in clockwise ordering ofter b. Not both a' and b' can be on the y-axis, otherwise P would not be a convex polygon. If one of these points is on the y-axis, we can assume without loss of generality that it is b'. In which case, $b' = a$. Consider the angle α formed by the line segment $\overline{aa'}$ and the line L where $0 < \alpha < 90^o$. Also, consider the angle β formed by the line segment $\overline{bb'}$ and the x-axis where $0 < \beta \leq 90^o$. Again, without loss of generality we can assume $0 < \alpha \leq \beta \leq 90^o$, since we could accomplish this relationship of the angles merely by changing the labels for the points a and b. We label the vertices of $P, P_1, P_2, P_3, \ldots, P_n$ where $P_1 = a$, $P_2 = a'$ and we continue the labeling in this successive clockwise manner. The sides are labeled $L_1, L_2, \ldots, L_n$ where $L_i = \overline{P_i P_{i+1}}$ for $i = 1, 2, ..., n-1$ and $L_n = \overline{P_n P_1}$. For $i = 1, 2, ..., n$ let R_i be the line containing the side L_i. Because of a previous assumption, we know that only two of the vertices P_i and P_{i+1} lie on R_i. Let A_i for $i \in \{1, 2, ..., n\}$ be a vertex of P such that $dist\,[R_i, A_i] \geq dist\,[R_i, A]$ for each $A \in P$. Note that for some $k \in \{2, 3, ..., n\}$ $P_k = b$ and $dist\,(P_k, R_1) \geq dist\,(A, R_1)$ for all $A \in P$, so we let $A_1 = P_k = b$.

We will describe the motion of two points F and G tracing P. Both points will move only in the clockwise direction and each point will stop when it reaches the starting position of the other point. When F travels along one side of P, G will remain still at one of the vertices of P and vice versa. Except for some slight modifications, the movements of these points F and G are the same as was described by K. Tkaczyńska in [T1].

The point F begins at $P_1 = a$ and G at $A_1 = P_k = b$.

Step 1 G remains at P_k, while F travels along L_1 until it reaches P_2. Notice that for each $x \in L_1$

$$d(A_1, x) \geq d(A_1, L_1) \geq d(A_1, R_1).$$

Consider the lines R_2 and R_k, containing L_2 and L_k respectively. There are two cases:

case 1: $R_2 \cap R_k \neq \emptyset$.

case 2: $R_2 \cap R_k = \emptyset$.

case 1: $R_2 \cap R_k \neq \emptyset$.

Since P is convex, it is contained in one of the four infinite wedges formed by R_2 and R_k. Consider the clockwise motion of the points F and G. One of the points would move toward the point of intersection of R_2 and R_k and the other would move away from this point of intersection. We let the point, that would move away from this point of intersection, move while the other print remains fixed.

case 2: $R_2 \cap R_k = \emptyset$.

We arbitrarily choose one of the points, either F or G, to move along the succeeding side of P, while the other point remains fixed.

After we complete step 2, F is at P_i and G is at P_j where either $i = 2$ and $j = k + 1$ or $i = 3$ and $j = k$.

Notice that in our very first step, we did follow the pattern for movement given in step 2. In case $R_1 \cap R_k = \emptyset$ (i.e. $\alpha = \beta$), we let G remain at P_k while F moves along L_1 to P_2. In case $R_1 \cap R_k \neq \emptyset$ ($\alpha < \beta$), again G remains fixed at P_k while F moves along L_1 and away from the point of intersection of R_1 and R_k.

We now consider the lines R_i and R_j containing L_i and L_j respectively and repeat the procedure described in step 2. At each step, one point stays fixed while the other point advances one side. Clearly, after n steps a total of n sides have been covered by F and G together. We now want to show that after n steps F is at P_k (F covered $k-1$ sides of P) and G is at P_1 (G covered $n-k+1$ sides of P).

It is clear that in these n steps there is an l, where $1 \leq l \leq n$ such that on step l either

case 1: F is at P_k and G is at P_j where $k < j \leq n$ or $j = 1$.

or

case 2: G is at P_1 and F is at j where $1 < j \leq k$.

First we consider case 1. If F is at P_k and G is at P_1, then F has covered $k-1$ sides, G has covered $n+1-k$ sides and $l = k-1+n+1-k = n$. So we have our desired result. Suppose F is at P_k and G is at P_j where $k < j \leq n$. Then F has covered $k-1$ sides and G has covered $j-k$ sides so $l = k-1+j-k = j-1$. We claim, that according to our algorithm, on the steps $l+1 = j$ to n, F remains at P_k while G covers the sides L_j through L_n.

We claim, that if at a step in the movements of F and G, F is at P_k and G is at P_j where $k < j \leq n$, then on the next step F remains at P_k while G advances to P_{j+1}. In the case $j = k+1$, it is clear that in the next step F remains at P_k while G advances to $P_{j+1} = P_{k+2}$. So, we just need to consider the situation when $j \geq k+2$. Let β be the angle formed by L_k and the x-axis where $0 < \beta \leq 90^o$.

case A: $\beta = 90^o$

In this case $P_j \in \overline{P_k P_1}$. But this can not happen because no three vertices of P lie on are straight line. So, $0 < \beta < 90^o$.

case B: $0 < \beta < 90^o$

In this case P_j must be contained in the triangle, T, bounded by the line R_k, the line L (i.e. the line parallel to the axis through the vertex P_1) and the line segment, S, with endpoints P_{k+1} and P_1. Also, $P_j \notin L$. Let θ be the angle formed by the line segment S and the ray R, starting at P_{k+1} through the vertex P_j, where θ is measured starting at S and in the counterclockwise direction. So $0 < \theta$, since $P_j \notin S$. Then P_{j+1} must be contained in the triangle, T', bounded by the line segment, S', which joins

P_j and P_1, the line L, and the ray R. This means that R_j and R_k intersect at a point p such that P_{k+1} is between p and P_k on the line R_k. According to our algorithm F must remain at P_k while G advances from P_j to P_{j+1}.

Now we consider case 2. If G is at the vertex P_1 and F is at P_k, then $l = n+1-k+k-1 = n$ and we have our desired result. Suppose G is at P_1 and F is at P_j where $1 < j < k$, then G has covered $n+1-k$ sides, F has covered $j-1$ sides and $l = n+1-k+j-1 = n+j-k$. We claim that, according to our algorithm, in the next $k-j$ steps, G remains at P_1 and F covers sides L_j through L_{k-1}. The proof of this is comparable to the one given in case 1.

So, in both cases we see that after n steps F is at the vertex P_k after having covered sides L_1 through L_{k-1} and G is at the vertex P_1 after having covered sides L_k through L_n.

Based on these n steps, it is clear that we can define steps for each positive integer. In the first n steps, F moves in clockwise order from P_1 to P_k and G moves in clockwise order from P_k to P_1. In steps $n+1$ to $2n$, G moves from P_1 to P_k in clockwise order and F moves from P_k to P_1 in clockwise order. In this same manner we can define the steps for all positive integers. We can define steps on the negative integers in a similar manner, but where F and G move counter clockwise. The point F starts at P_1 and G starts at P_k. In steps -1 to $-n$, F moves (counter clockwise) to P_k and G moves counterclockwise to P_1. During the next n-steps ($-n-1$ to $-2n$), F would return to P_1 and G would return to P_k, both moving in the counterclockwise direction. In the same manner we can define steps for all Z^-.

We claim that whenever a point travels along a side L_i while the other point remains at a vertex P_j then

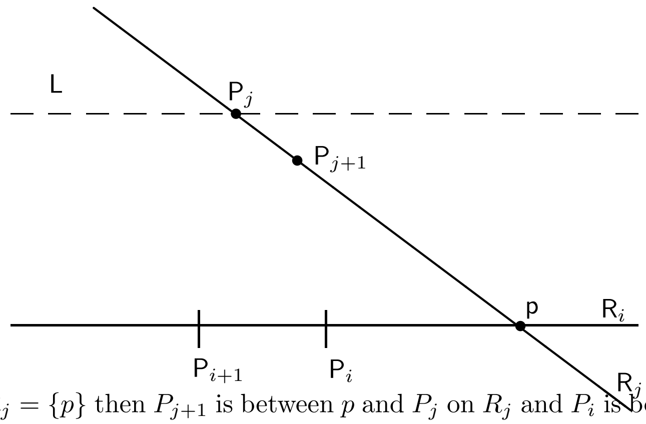
$$dist[R_i, P_j] = dist[R_i, A_i].$$

Suppose F remains at P_j while G travels from P_i to P_{i+1}. There are two cases to consider

Case 1 $R_i \cap R_j = \emptyset$

In this case R_i and R_j are parallel and P is contained in the portion of the plane bound by the lines R_i and R_j and on the lines R_i and R_j. It is clear that $dist[R_i, R_j] = dist[R_i, A_i]$ and that $A_i = P_j$ or $A_i = P_{j+1}$.

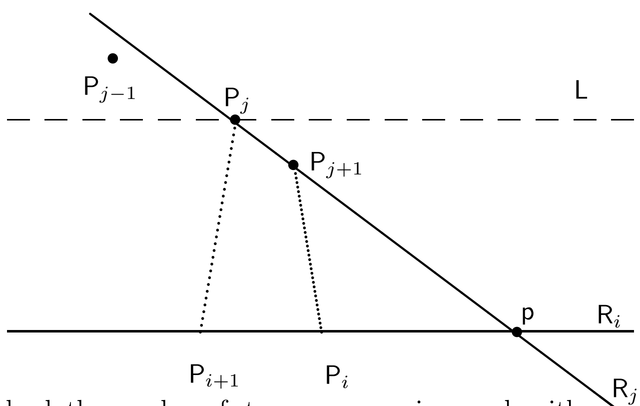
Case 2 $R_i \cap R_j \neq \emptyset$



Let $R_i \cap R_j = \{p\}$ then P_{j+1} is between p and P_j on R_j and P_i is between p and P_{i+1} on R_i. This is true since our labeling is in clockwise order and based on our algorithm for F and G. Let L be the line parallel to R_i, through the vertex P_j. The vertex P_{j-1} must be contained in the wedge W formed by R_j and R_i which contains all of P. If $P_{j-1} \in L$, then

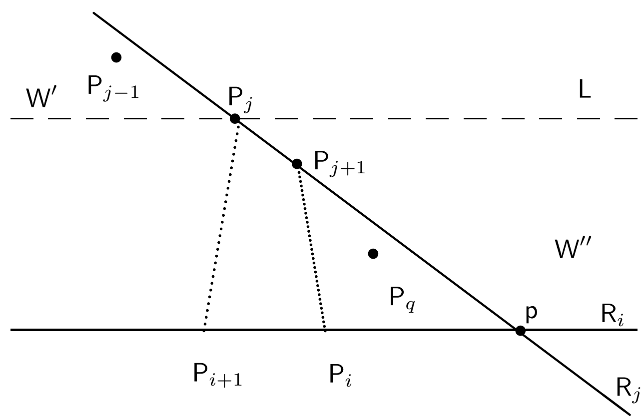
$$dist(R_i, P_j) = dist(R_i, P_{j-1}) = dist(R_i, A_i)$$

and either $P_j = A_i$ or $P_{j-1} = A_i$. If P_{j-1} is in the portion of the plane bound by L and R_i, then it is clear that $dist[R_i, P_j] = dist[R_i, A_i]$ and $P_j = A_i$. The other possibility is that P_{j-1} is in the wedge W' formed by R_j and L which is in W, but not in the portion of the plane bound by L and R_i. Also $P_{j-1} \notin (L \cup R_j)$.



We can go back the number of steps necessary in our algorithm so that F is at P_{j-1} (that is F moves back exactly one vertex), and G is at P_q where P_q is between P_j and P_i on P, but not on the portion of P that contains

P_{i+1}. If $P_q = P_j$ then on the next step F would remain at P_{j-1} while G moves to P_{j+1}. If $P_q = P_i$ then on the next step F would remain at P_{j-1} and G would move to P_{i+1}. So we can assume that $P_q = P_{j+1}$ or $P_q = P_{i-1}$ or P_q is between P_{j+1} and P_{i-1} on P but not on the portion that contains P_{i+1}. Then P_q must be in the triangle T bound by R_i, R_j and $\overline{P_iP_{j+1}}$. Note, P_q is not on the boundary of the triangle T. The line R_{j-1} is contained in $W' \cup W''$ where W'' is the wedge formed by L and R_j which is opposite to the wedge W'. We now consider how R_{j-1} and R_q are related. The lines R_q and R_{j-1} are not parallel, since P_i would not be contained between them. Also, R_q and R_{j-1} can not intersect in W' since again this would exclude P_i as an element of P. So, R_{j-1} and R_q must intersect in W''. According to our algorithm and since labeling of the vertices are clockwise, in the next step, F remains at P_{j-1} and G moves from P_q to the next vertex towards P_i. In each succeeding step, F would remain at P_{j-1} until G reaches P_i. Then again, by our algorithm, F remains at P_{j-1} and G moves to P_{i+1}. Hence $P_{j-1} \notin W'$. So, we get our desired result. That is if F stays at P_j while G moves along L_i, then $dist(R_i, P_j) = dist(R_i, A_i)$ and vice versa.



Let $a_i = dist\,[A_i, R_i]$ and let $a = \min\{a_i|\ i = 1, \ldots, n\}$. The motion of the points F and G are such that the distance between them is always larger than or equal to a. Tkaczyńska has shown in [T1] that $\sigma(P) = a$, so the minimum distance between them can not be bigger than a.

In this section we use P_{n+1} as a second labeling for the vertex P_1.

The movements of F and G determine two increasing functions f and g. The function f is defined as follows

$$f : \{1, 2, \dots, k\} \to \{k, \dots, n, n+1\}$$

given by $f(1) = l_1 = k$ and $f(j) = l_j$ where P_{l_j} is a vertex of P such that F is at the vertex P_j and G is at the vertex P_{l_j} and l_j is the largest index for which this is true. By a previous observation, we see that $f(1) = k$ and $f(k) = l_k = n + 1$. The function g is defined as follows

$$g : \{k, k+1, \dots, n, n+1\} \to \{1, 2, \dots, k\}$$

given by $g(j) = l_j$ where P_{l_j} is a vertex of P such that G is at the vertex P_j and F is at the vertex P_{l_j} and l_j is the largest such index. By previous observation we see that $g(k) = l_k \geq 2$ and $g(n + 1) = l_{n+1} = k$.

We now construct a new convex polygon Q, where the vertices of Q are a subset of the vertices of P. According to our algorithm and functions f and g, when the point F is at the vertex $P_{g(k)}$ the point G moves along the sides of P from the vertex P_k to the vertex $P_{f \circ g(k)}$. We replace these sides with a new side connecting vertices P_k and $P_{f \circ g(k)}$. This side corresponds to an old side of P if $f \circ g(k) = k + 1$. Now G is at the vertex $P_{f \circ g(k)}$ and F is at the vertex $P_{g(k)}$. Now, G stays at the vertex $P_{f \circ g(k)}$ while F travels along the sides of P from the vertex $P_{g(k)}$ to the vertex $P_{g \circ f \circ g(k)}$. We replace the sides of P covered by F with a side connecting vertices $P_{g(k)}$ and $P_{g \circ f \circ g(k)}$. Again, this corresponds to a side of P if $g \circ f \circ g(k) = g(k) + 1$.

We continue this process until the last two sides have been constructed, that is the side connecting P_j and P_{n+1} where $k < j \leq n$ and the side connecting P_m and P_k where $1 < m \leq k - 1$.

We now make one final replacement. If the last side constructed was the side connecting P_m and P_k where $1 < m \leq k-1$ (i.e. the point G stayed at P_{n+1} while F moved from the vertex P_m to P_k) then replace the sides L_1 through $L_{g(k)-1}$ with a side connecting the vertices P_1 and $P_{g(k)}$. If the last side constructed was the side connecting P_j and P_{n+1} where $k < j \leq n$ (i.e. the point F stayed at P_k while G moved from P_j to P_{n+1}), then replace this side with a side connecting P_j and $P_{g(k)}$.

We can see from this construction that each vertex of Q is paired with exactly one side of Q. We can show also that the number of vertices of Q is odd. Draw a line L^* through P_k and the midpoint of the side corresponding to it, the side connecting P_j and $P_{g(k)}$ (either $j = 1$ or $k < j \leq n + 1$). Consider the two components of $Q - L^*$. Suppose there are q vertices in the component containing

$P_{g(k)}$ then in the other component there must be q sides corresponding to these vertices. Consequently there are also q vertices in the other component. Hence the total number of vertices of Q is $m = 2q+1$. We can label these vertices. We let $P_j = B_1$ where either $k < j \le n$ or $j = 1$ and $\overline{P_j P_{g(k)}}$ is a side of Q. We continue labeling in the counterclockwise order. Hence, $P_{g(k)} = Bm$ and $P_k = B_{q+1}$. So B_1 corresponds to the side connecting B_{q+1} and B_{q+2}. In general each vertex B_i corresponds to the side $B_{i+\frac{m-1}{2}}\ B_{i+\frac{m+1}{2}}$, with subscripts taken modulo m.

Let O be a point in the bounded complement of Q. We rotate the plane clockwise through the smallest possible angle so that the ray $\overrightarrow{OB_1}$ coincides with the positive x-axis. Let θ_j for $j = 1, 2, \cdots, m$ be the angle that the ray $\overrightarrow{OB_j}$ makes with the positive x-axis, where the angle is measured in the counter-clockwise direction. Clearly, $0 = \theta_1 < \theta_2 < \ldots < \theta_m < 2\pi$.

We pick an angle α, such that $0 < \alpha < \frac{1}{4}\min\{\theta_{j+1} - \theta_j$ for $j = 1, 2, \cdots, m-1$; $2\pi - \theta_m\}$ and so that if $re^{i\theta} \in Q$ where $\theta_j - \alpha < \theta < \theta_j + \alpha$, then $\rho(re^{i\theta}, B_j) < \frac{\epsilon}{2}$.

Let W_j be the portion of the plane which is bounded by the rays $\overrightarrow{Oe^{i(\theta_j-\alpha)}}$ and $\overrightarrow{Oe^{i(\theta_j+\alpha)}}$ and contains the point B_j. Let W'_j be the portion of the plane bounded by the rays $\overrightarrow{Oe^{i(\theta_{j+(m-1)/2}+\alpha)}}$ and $\overrightarrow{Oe^{i(\theta_{j+(m+1)/2}-\alpha)}}$ which does not contain the point B_j. If $p \in W_j \cap X$ and $q \in W'_j \cap X$ then $\rho(p, q) \ge \rho(B_j, P \cap W'_j) - \frac{\epsilon}{2}$, since $\rho(B_j, P \cap W'_j) - \frac{\epsilon}{2} \ge b(P) - \frac{\epsilon}{2}$ and $b(P) - \frac{\epsilon}{2} \ge b(\partial C) - \frac{3\epsilon}{2}$. Consequently, $\rho(p, q) \ge b(\partial C) - \frac{3\epsilon}{2} = \sigma(\partial C) - \frac{3\epsilon}{2}$.

Let S be the unit circle centered at O. Let $p : X \longrightarrow S$ be the map defined by $p(re^{i\theta}) = e^{i\theta}$. Clearly, p is essential. Let q be any 1-1 map on S such that

$$q(e^{i\theta}) = \begin{cases} e^{i\frac{2\pi}{m}(j-1)}, & \text{for } \theta = \theta_j,\ j = 1, 2, \cdots, m \\ e^{i(\frac{2\pi}{m}(j-1) - \frac{2\pi}{4m})}, & \text{for } \theta = \theta_j - \alpha,\ j = 1, 2, \cdots, m \\ e^{i(\frac{2\pi}{m}(j-1) + \frac{2\pi}{4m})}, & \text{for } \theta = \theta_j + \alpha,\ j = 1, 2, \cdots, m \end{cases}$$

Also, the map $q \circ p : X \longrightarrow S$ is essential. We can see that if $q \circ p(x) = s$ and $q \circ p(y) = -s$, then one of the points, say x, must be contained in W_j and the other point, y, must be contained in W'_j. Consequently, $\inf_{s\in S} \rho((q \circ p)^{-1}(s), (q \circ p)^{-1}(-s)) \ge b(\partial C) - \frac{3\epsilon}{2}$. Since this is true for all ϵ such that $0 < 3\epsilon < b(\partial C)/2$, $\inf_{s\in S} \rho((q \circ p)^{-1}(s), (q \circ p)^{-1}(-s)) \ge b(\partial C) = \sigma(\partial C)$. By Theorem L, $\sigma(\partial C) \le \sigma(X)$. □

Corollary 2.1. *Let X be a separating planar continuum and let C be a convex region contained in a bounded component of the complement of X. Then $\sigma_0(X) \ge \sigma_0(\partial C)$.*

Proof. This is true since, $\sigma_0(X) \ge \sigma(X)$ and $\sigma_0(\partial C) = \sigma(\partial C) = b(\partial C)$. □

Corollary 2.2. *Let X be a simple closed curve in the plane and let C be a convex region contained in the bounded component of the complement of X. Then $\tau(X) \geq \tau(\partial C)$ where $\tau = \sigma$, σ_0, σ^* , or σ_0^*.*

Proof. This is true since when X is a simple closed curve, $\sigma(X) = \sigma^*(X)$ and $\sigma_0(X) = \sigma_0^*(X)$. □

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