High-Order Finite Difference Methods for Hyperbolic IBVP

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Abstract

We have derived stability results for explicit high-order finite difference approximations of systems of hyperbolic initial-boundary value problems (IBVP). The schemes are a generalization of a fourth order scheme by Gustafsson, Kreiss and Oliger [5] to general order of accuracy $2\,r$. The stability results are obtained using the theory of Gustafsson, Kreiss and Sundström (G-K-S) for the semi-discrete IBVP. These results are then generalized to the fully discrete case using a theory of Kreiss and Wu [7].

Key words: finite differences, wave equation, stability analysis.

AMS subject classifications: 65M06, 65M12.

1 Introduction

In this paper we develop explicit high-order difference methods for hyperbolic systems and use them in practical computation of the two-dimensional wave equation. For a hyperbolic system to preserve the spatial accuracy, a pth-order inner scheme must be closed with at least a $(p-1)$th-order boundary scheme, see Gustafsson [3] and [4].

When investigating stability of the numerical approximation of the IBVP, we rely on the stability theory analysis based on normal mode analysis, developed for the fully discrete case by Gustafsson, Kreiss and Sundström (G-K-S) [6], and for the semi-discrete case by Strikwerda [10] and Gustafsson, Kreiss and Oliger [5]. The G-K-S theory gives conditions that the inner and boundary schemes must satisfy to ensure stability. The following theorem states when hyperbolic systems are G-K-S stable.

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Theorem 1.1 *Necessary and sufficient conditions for stability (fully discrete [6] or semi-discrete [5],[10]) of the finite-domain IBVP is that, the inner scheme must be Cauchy stable on $(-\infty, \infty)$, and that the Kreiss condition is fulfilled, i.e. there are no eigensolutions for the two quarter-plane problems.*

Furthermore, in [5] it is shown that if the conditions in theorem 1.1 are fulfilled, the normal mode analysis leads to strong stability.

Here, we will use the method of lines approach, the hyperbolic systems are discretized in space but the time is left continuous. The semi-discrete system is then analyzed and stability results derived. The stability of the fully discrete problem follows from a result of Kreiss and Wu [7]. They have shown that under weak conditions, if specific Runge-Kutta or multi-step schemes are used for time-integration, the stability of the fully discrete problem follows from the stability of the semi-discrete problem.

Explicit difference operators for PDEs have been considered, for example in [1], [2], [5] and [9]. In [5] strong stability for hyperbolic systems in one dimension is shown for the fourth order case. In this paper we generalize the result to general order of accuracy $2\,r$. The organization of the paper is as follows. Section 2 presents the stability analysis on the scalar model problem $u_t = a\,u_x$. In section 3 the result is generalized to systems, and in section 4 numerical results on the two-dimensional wave equation are presented.

2 Scalar model equation for IBVP

Consider the problem

$$(1) \quad \frac{\partial\,u(x,t)}{\partial\,t} = a\,\frac{\partial\,u(x,t)}{\partial\,x}, \quad a \neq 0, \ 0 \leq x < \infty, \ t \geq 0,$$

$$(2) \quad u(x,0) = f(x),$$

$$(3) \quad u(0,t) = g(t), \quad \text{if } a < 0.$$

We want to solve the above problem by difference approximation. Therefore, we introduce the mesh width h, and divide the x-axis into intervals of length h. For $j = -r + 1, -r + 2, \ldots, 0, 1, \ldots$ we use the notation

$$v_j(t) = v(x_j, t), \quad x_j = j h.$$

We approximate (1) for $j = 1, 2, \ldots$ by a centered finite difference scheme of order $2r$

$$
(4) \qquad
\begin{aligned}
\frac{\partial v_j(t)}{\partial t} &= aQv_j(t), \\
v_j(0) &= f(x_j),
\end{aligned}
$$

where

$$(5) \qquad Q = \tfrac{1}{h} \sum_{\nu=-r}^{r} \alpha_\nu E^\nu, \quad E^\nu v_j = v_{j+\nu},$$

is difference operator and the coefficients are given by

$$
(6) \qquad
\begin{aligned}
\alpha_\nu &= \frac{(-1)^{\nu-1}(r!)^2}{\nu(r+\nu)!(r-\nu)!}, \quad \nu = 1, \ldots, r, \\
\alpha_{-\nu} &= -\alpha_\nu, \qquad\qquad \nu = 0, \ldots, r.
\end{aligned}
$$

Because the operator is $2r + 1$ points wide we need extra boundary conditions at points $x_{-\nu}$, $\nu = 0, 1, \ldots, r - 1$. If $a > 0$, we have outflow at $x = 0$, and use extrapolation of order q

$$(7) \qquad (hD_+)^q v_{-\nu}(t) = 0, \quad \nu = 0, 1, \ldots, r - 1.$$

If $a < 0$, $x = 0$ is an inflow boundary, and by differentiating the boundary condition $u(0, t) = g(t)$ and using the differential equation we obtain

$$(8) \qquad a^{2\nu} \frac{\partial^{2\nu} u(0,t)}{\partial x^{2\nu}} = g^{(2\nu)}(t), \quad \nu = 0, 1, \ldots.$$

We need

Lemma 2.1 *We have the following expansion for smooth functions $u(x)$*

$$(9) \quad (D_+D_-)^\nu u(0) = u^{(2\nu)}(0) + \sum_{j=1}^{\infty} c_j^\nu h^{2j} u^{(2\nu+2j)}(0),$$

where $h\,D_+ = E - I$ and $h\,D_- = I - E^{-1}$. The coefficients are defined as

$$
\begin{aligned}
c_j^\nu &= \frac{1}{(2(\nu+j))!} \sum_{k=0}^{2\nu} \binom{2\nu}{k} (-1)^k (\nu - k)^{2(\nu+j)}, \\
& \qquad j = 0, 1, \ldots, \ \nu = 0, 1, \ldots.
\end{aligned}
$$

Proof

$(D_+D_-)^\nu u(0)$

$$
\begin{aligned}
&= (D_+D_-)^{\nu-1} \sum_{j_1=1}^{\infty} 2\frac{h^{2j_1-2}}{(2j_1)!} u^{(2j_1)}(0) \\
&= \sum_{j_1=1}^{\infty} 2\frac{h^{2j_1-2}}{(2j_1)!} (D_+D_-)^{\nu-1} u^{(2j_1)}(0) \\
&= \sum_{j_1=1}^{\infty} 2\frac{h^{2j_1-2}}{(2j_1)!} \sum_{j_2=1}^{\infty} 2\frac{h^{2j_2-2}}{(2j_2)!} \cdots \\
&\qquad \sum_{j_\nu=1}^{\infty} 2\frac{h^{2j_\nu-2}}{(2j_\nu)!} u^{(2(j_1+\ldots+j_\nu))}(0) \\
&= \sum_{j_1,\ldots,j_\nu}^{\infty} 2^\nu \frac{h^{2(j_1+\ldots+j_\nu)-2\nu}}{(2j_1)!\cdots(2j_\nu)!} u^{(2(j_1+\ldots+j_\nu))}(0).
\end{aligned}
$$

This shows that $(D_+D_-)^\nu$ has an expansion of the form (9). To compute the coefficients c_j^ν we note that

$$
\begin{aligned}
(D_+D_-)^\nu &= \tfrac{1}{h^{2\nu}}(E - I)^{2\nu} E^{-\nu} \\
&= \tfrac{1}{h^{2\nu}} \sum_{k=0}^{2\nu} \binom{2\nu}{k} E^{\nu-k}(-I)^k
\end{aligned}
$$

where the binomial theorem have been used. Thus,

$$(D_+D_-)^\nu u(0) = \frac{1}{h^{2\nu}} \sum_{k=0}^{2\nu} \binom{2\nu}{k} (-1)^k u(x_{\nu-k}).$$

Expanding $u(x_{\nu-k})$ in Taylor series implies

$$(D_+D_-)^\nu u(0) = \sum_{l=0}^{\infty} h^{l-2\nu} \frac{u^{(l)}(0)}{l!} \sum_{k=0}^{2\nu} \binom{2\nu}{k} (-1)^k (\nu - k)^l.$$

The coefficient c_j^ν is then obtained for $l = 2(\nu + j)$ and the proof if complete . $\qquad\square$

As boundary conditions for the difference approximation in the inflow case we approximate (8) for $\nu = 0, \ldots, r - 1$ by

$$(10) \quad a^{2\nu}(D_+D_-)^\nu v_0(t) = \sum_{j=0}^{r-\nu-1} c_j^\nu h^{2j} \frac{g^{(2\nu+2j)}(t)}{a^{2j}}.$$

For $\nu = 0$ the coefficients are $c_j^0 = \delta_{j,0}$, and (10) is valid also for the analytic boundary condition.

2.1 Necessary and sufficient conditions for stability

A necessary condition for stability of our semi-discrete approximation, defined in (4), (7) and (10), is that the associated eigenvalue problem has no eigenvalues or generalized eigenvalues. That is, our semi-discrete problem with

$g = 0$, $f = 0$, has no solutions of the form

$$(11) \qquad v_j(t) = e^{st} \varphi_j,$$

where s are the eigenvalues. Substitution into (4), (7) and (10) yields the eigenvalue problem

$$(12) \quad \tilde{s}\varphi_j = a \sum_{\nu=-r}^{r} \alpha_\nu \varphi_{j+\nu}, \quad j = 1, 2, \ldots, \quad \tilde{s} = sh,$$

$$(13) \quad D_+^q \varphi_{-\nu} = 0, \quad \nu = 0, \ldots, r-1, \text{ if } a > 0,$$

$$(14) \quad (D_+ D_-)^\nu \varphi_0 = 0, \quad \nu = 0, \ldots, r-1, \text{ if } a < 0,$$

$$(15) \quad \|\varphi\|_h < \infty.$$

The characteristic equation to the difference equation (12) is

$$(16) \qquad \tilde{s} = a \sum_{\nu=1}^{r} \alpha_\nu \left(\kappa^\nu - \kappa^{-\nu} \right).$$

Scalar product and norm are defined by

$$(17) \qquad (v, w)_h = \sum_{j=1}^{\infty} v_j \bar{w}_j h, \quad \|v\|_h = (v, v)_h.$$

First we note that in [5] it is shown that for sufficiently large $|\tilde{s}|$, $\text{Re } \tilde{s} > 0$, there are no eigenvalues. Furthermore, when $\text{Re } \tilde{s} \to \infty$, the κ with $|\kappa| < 1$ converges to zero. We need

Lemma 2.2 *1) The characteristic equation (16) has exactly r roots*

$$|\kappa_j| < 1 \text{ for } \text{Re } \tilde{s} > 0, \quad j = 1, \ldots, r,$$

and there are no roots with $|\kappa_j| = 1$ for $\text{Re } \tilde{s} > 0$.
2) For $\tilde{s} = 0$ the only roots to the characteristic equation (16) with absolute value one are $\kappa = \pm 1$. Furthermore they are simple.
3) In a neighborhood of $\tilde{s} = 0$ the roots with $|\kappa_j| < 1$ for $\text{Re } \tilde{s} > 0$, and absolute value one for $\tilde{s} = 0$, are of the form

$$\kappa_1 = -1 + \frac{\tilde{s}}{2a \sum_{k=1}^{r} \frac{(r!)^2}{(r+k)!(r-k)!}} + O(\tilde{s}^2), \quad \text{if } a > 0,$$

$$\kappa_1 = 1 + \frac{\tilde{s}}{a} + O(\tilde{s}^2), \qquad\qquad \text{if } a < 0.$$

Proof
1) The statement follows from a result in [5].
2) Also in [5] it is shown that the operator can be factorized as

$$Q = D_0 \sum_{\nu=0}^{r-1} (-1)^\nu \beta_\nu (h^2 D_+ D_-)^\nu,$$

where the coefficients are defined recursively by

$$\beta_\nu = \frac{\nu}{4\nu+2} \beta_{\nu-1}, \quad \nu = 1, \ldots, r-1,$$

$$\beta_0 = 1.$$

Since $(h^2 D_+ D_-)^\nu \kappa^j = \kappa^j \frac{(\kappa-1)^{2\nu}}{\kappa^\nu}$, the characteristic equation can be written as

$$\tilde{s} = a \frac{1}{2} (\kappa - \kappa^{-1}) \sum_{\nu=0}^{r-1} \beta_\nu (-1)^\nu \frac{(\kappa-1)^{2\nu}}{\kappa^\nu}.$$

Let $\tilde{s} = 0$, $\kappa = e^{i\xi}$, $-\pi \leq \xi \leq \pi$, and note that

$$\left(\frac{(e^{i\xi}-1)^2}{e^{i\xi}} \right)^\nu = (-1)^\nu 2^{2\nu} (\sin \frac{\xi}{2})^{2\nu},$$

then the condition to have a root on the unit circle is

$$a \, i \sin \xi \sum_{\nu=0}^{r-1} \beta_\nu 2^{2\nu} (\sin \frac{\xi}{2})^{2\nu} = 0.$$

The second factor is positive for all ξ except $\xi = 0$ for which it is zero. Therefore the condition can be fulfilled only if $\xi = 0$ or $\xi = \pm\pi$, i.e, if $\kappa = \pm 1$. To show that these roots are simple, let $p(\kappa) = \sum_{\nu=1}^{r} \alpha_\nu (\kappa^\nu - \kappa^{-\nu})$. A necessary condition for $\kappa = \pm 1$ to be a multiple root is $dp(\pm 1)/d\kappa = 0$, but

$$\frac{dp(\kappa)}{d\kappa} = \sum_{\nu=1}^{r} \alpha_\nu \nu (\kappa^\nu + \kappa^{-\nu}) \kappa^{-1},$$

$$\frac{dp(1)}{d\kappa} = 2 \sum_{\nu=1}^{r} \alpha_\nu \nu = 1,$$

$$\frac{dp(-1)}{d\kappa} = -2 \sum_{\nu=1}^{r} \alpha_\nu \nu (-1)^\nu$$

$$= 2 \sum_{\nu=1}^{r} \frac{(r!)^2}{(r+\nu)!(r-\nu)!} \neq 0,$$

shows that $\kappa = \pm 1$ are simple roots.
3) For $\tilde{s} = 0$, the solutions to the characteristic equation (16) with absolute value one are

$$\kappa^{(1,2)} = \pm 1.$$

Since,

$$(1 + \epsilon)^k = 1 + \epsilon k + O(\epsilon^2),$$

and

$$(-1 + \epsilon)^k = (-1)^k (1 - \epsilon k) + O(\epsilon^2),$$

the characteristic equation (16) gives for small $\tilde{s}$ and $\kappa^{(1)} = 1 + \epsilon$

$$\tilde{s} = a \sum_{k=1}^{r} \alpha_k \left((1+\epsilon)^k - (1+\epsilon)^{-k} \right) = 2 a \epsilon \sum_{k=1}^{r} \alpha_k k + O(\epsilon^2).$$

Consistency implies that $\sum_{k=1}^{r} \alpha_k\, k = 1/2$, therefore

$$\kappa^{(1)} = 1 + \frac{\tilde{s}}{a} + O(\tilde{s}^2).$$

With $\kappa^{(2)} = -1 + \epsilon$ we have

$$
\begin{aligned}
\tilde{s} &= 2\,a\,\epsilon \sum_{k=1}^{r} \alpha_k (-1)^{k+1} k + O(\epsilon^2) \\
&= 2\,a\,\epsilon \sum_{k=1}^{r} \frac{(r!)^2}{(r+k)!(r-k)!} + O(\epsilon^2)
\end{aligned}
$$

where $\sum_{k=1}^{r} \frac{(r!)^2}{(r+k)!(r-k)!}$ is positive. Thus,

$$\kappa^{(2)} = -1 + \frac{\tilde{s}}{2a \sum\limits_{k=1}^{r} \frac{(r!)^2}{(r+k)!(r-k)!}} + O(\tilde{s}^2).$$

By selecting the roots satisfying $|\kappa| < 1$ for $Re\ \tilde{s} > 0$, the third statement follows and the lemma is complete. $\square$

By Lemma 1.2, there are r roots κ_ν, $\nu = 1, \ldots, r$, with $|\kappa_\nu| < 1$ for $Re\ \tilde{s} > 0$. The general solution of (12) with $\|\varphi\|_h < \infty$ can be written in the form

$$
\begin{aligned}
\varphi_j &= c_1 f_j[\kappa_1] + c_2 f_j[\kappa_2, \kappa_1] + \ldots + c_r f_j[\kappa_r, \ldots, \kappa_1], \\
& \kappa_i \neq \kappa_k,\ i \neq k,\ j = 1, 2, \ldots,
\end{aligned}
$$
(18)

where

$$f_j[\kappa] = \kappa^j.$$

$$f_j[\kappa_l, \ldots, \kappa_k] = \frac{f_j[\kappa_l, \ldots, \kappa_{k+1}] - f_j[\kappa_{l-1}, \ldots, \kappa_k]}{\kappa_l - \kappa_k}, \quad l > k.$$

If for instance $\kappa_1 = \kappa_2 = \kappa$ is a double root, then $f_j[\kappa_2, \kappa_1]$ becomes $j\kappa^{j-1}$. If $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$ is a triple root then $f_j[\kappa_3, \kappa_2, \kappa_1]$ will become $j(j-1)\kappa^{j-2}$. If $\kappa_1 = \kappa_2 = \ldots = \kappa_l = \kappa$ is a root with multiplicity l, then $f_j[\kappa_l, \ldots, \kappa_1]$ becomes $\frac{j!}{(j-(l-1))!}\kappa^{j-(l-1)}$, and the solution (18) to the eigenvalue problem in this case can then be written as

$$\varphi_j = (\tilde{c}_1 + \tilde{c}_2 j + \ldots + \tilde{c}_l j^{l-1})\kappa^j + \sum_{\nu=l+1}^{r} \tilde{c}_\nu f_j[\kappa_\nu, \ldots, \kappa_{l+1}].$$

Therefore by using the form (18) we can treat simple roots and multiple roots simultaneously.

To be able to express the boundary conditions in terms of κ we need the following relations for the difference operators D_+ and D_-

$$
\begin{aligned}
(hD_+)^q \kappa^j &= (hD_+)^{q-1}(\kappa^{j+1} - \kappa^j) \\
&= (hD_+)^{q-2}\kappa^j(\kappa - 1)^2 \\
&= \ldots = \kappa^j(\kappa - 1)^q, \\
(h^2 D_+ D_-)^\nu \kappa^j &= h^{2\nu} D_+^\nu D_-^\nu \kappa^j = h^{2\nu} D_+^{2\nu} \kappa^{j-\nu} \\
&= \kappa^{j-\nu}(\kappa - 1)^{2\nu}.
\end{aligned}
$$

By (18) the outflow boundary conditions becomes

$$
\begin{aligned}
(hD_+)^q \varphi_{-\nu} &= \sum_{k=1}^{r} c_k (hD_+)^q f_{-\nu}[\kappa_k, \ldots, \kappa_1] \\
&= \sum_{k=1}^{r} c_k g_\nu[\kappa_k, \ldots, \kappa_1] = 0, \\
& \nu = 0, 1, \ldots, r-1,
\end{aligned}
$$
(19)

with

$$g_\nu[\kappa] = \frac{(\kappa-1)^q}{\kappa^\nu},$$

$$g_\nu[\kappa_l, \ldots, \kappa_k] = \frac{g_\nu[\kappa_l, \ldots, \kappa_{k+1}] - g_\nu[\kappa_{l-1}, \ldots, \kappa_k]}{\kappa_l - \kappa_k}, \quad l > k.$$

The inflow boundary conditions becomes

$$
\begin{aligned}
(h^2 D_+ D_-)^\nu \varphi_0 &= \sum_{k=1}^{r} c_k (h^2 D_+ D_-)^\nu f_j[\kappa_k, \ldots, \kappa_1]\,\big|_{j=0} \\
&= \sum_{k=1}^{r} c_k g_\nu[\kappa_k, \ldots, \kappa_1] = 0, \\
& \nu = 0, 1, \ldots, r-1,
\end{aligned}
$$
(20)

where

$$g_\nu[\kappa] = \frac{(\kappa-1)^{2\nu}}{\kappa^\nu},$$

$$g_\nu[\kappa_l, \ldots, \kappa_k] = \frac{g_\nu[\kappa_l, \ldots, \kappa_{k+1}] - g_\nu[\kappa_{l-1}, \ldots, \kappa_k]}{\kappa_l - \kappa_k}, \quad l > k.$$

The systems of boundary conditions (19) and (20) we write as

$$D \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix} = 0,$$

where D is the $(r \times r)$-matrix

$$
\begin{pmatrix}
g_0[\kappa_1] & g_0[\kappa_2, \kappa_1] & \cdots & g_0[\kappa_r, \ldots, \kappa_1] \\
g_1[\kappa_1] & g_1[\kappa_2, \kappa_1] & \cdots & g_1[\kappa_r, \ldots, \kappa_1] \\
\vdots & & & \vdots \\
g_{r-1}[\kappa_1] & g_{r-1}[\kappa_2, \kappa_1] & \cdots & g_{r-1}[\kappa_r, \ldots, \kappa_1]
\end{pmatrix},
$$

and

$$g_\nu[\kappa] = \begin{cases} \dfrac{(\kappa-1)^{2\nu}}{\kappa^\nu}, & a < 0, \\[2mm] \dfrac{(\kappa-1)^q}{\kappa^\nu}, & a > 0, \end{cases}$$

$$g_\nu[\kappa_l, \ldots, \kappa_k] = \frac{g_\nu[\kappa_l, \ldots, \kappa_{k+1}] - g_\nu[\kappa_{l-1}, \ldots, \kappa_k]}{\kappa_l - \kappa_k}, \quad l > k.$$

The be able to calculate the determinant of D we use

Lemma 2.3 *D can be factorized as*

$$D = \tilde{D} \prod_{j=1}^{r-1} B_{r-j} C_{r-j}, \quad r > 1,$$

with

$$B_\nu = \begin{pmatrix} I_{r-(\nu+1)} & 0 \\ 0 & \tilde{B}_\nu \end{pmatrix},$$

$$\tilde{B}_\nu = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix},$$

$$C_\nu = \begin{pmatrix} I_{r-\nu} & 0 \\ 0 & \tilde{C}_\nu \end{pmatrix},$$

$$\tilde{C}_\nu = \begin{pmatrix} (\kappa_{r-(\nu-1)} - \kappa_1)^{-1} & & & \\ & (\kappa_{r-(\nu-2)} - \kappa_2)^{-1} & & \\ & & \ddots & \\ & & & \\ & & (\kappa_{r-1} - \kappa_{\nu-1})^{-1} & \\ & & & (\kappa_r - \kappa_\nu)^{-1} \end{pmatrix},$$

$$\tilde{D} = \begin{pmatrix} g_0[\kappa_1] & g_0[\kappa_2] & \cdots & g_0[\kappa_r] \\ \vdots & & & \vdots \\ g_{r-1}[\kappa_1] & g_{r-1}[\kappa_2] & \cdots & g_{r-1}[\kappa_r] \end{pmatrix}.$$

Here B_ν, C_ν, and $\tilde{D}$ are of size $(r \times r)$, $\tilde{B}_\nu$ is of size $((\nu+1) \times (\nu+1))$, $\tilde{C}_\nu$ is of size $(\nu \times \nu)$, and I_ν denotes the identity matrix of size $(\nu \times \nu)$.

Proof

$$D = \begin{pmatrix} g_0[\kappa_1] & g_0[\kappa_2, \kappa_1] & \cdots & g_0[\kappa_{r-1}, \ldots, \kappa_1] \\ \vdots & & & \\ g_{r-1}[\kappa_1] & g_{r-1}[\kappa_2, \kappa_1] & \cdots & g_{r-1}[\kappa_{r-1}, \ldots, \kappa_1] \end{pmatrix}$$

$$\frac{g_0[\kappa_r, \ldots, \kappa_2] - g_0[\kappa_{r-1}, \ldots, \kappa_1]}{\kappa_r - \kappa_1}$$
$$\vdots$$
$$\frac{g_{r-1}[\kappa_r, \ldots, \kappa_2] - g_{r-1}[\kappa_{r-1}, \ldots, \kappa_1]}{\kappa_r - \kappa_1}$$

$$= \begin{pmatrix} g_0[\kappa_1] & \cdots & g_0[\kappa_{r-1}, \ldots, \kappa_1] \\ \vdots & & \\ g_{r-1}[\kappa_1] & \cdots & g_{r-1}[\kappa_{r-1}, \ldots, \kappa_1] \end{pmatrix}$$

$$\begin{pmatrix} g_0[\kappa_r, \ldots, \kappa_2] \\ \vdots \\ g_{r-1}[\kappa_r, \ldots, \kappa_2] \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & (\kappa_r - \kappa_1)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} g_0[\kappa_1] & \cdots & g_0[\kappa_{r-2}, \ldots, \kappa_1] & g_0[\kappa_{r-1}, \ldots, \kappa_2] \\ \vdots & & & \\ g_{r-1}[\kappa_1] & \cdots & g_{r-1}[\kappa_{r-2}, \ldots, \kappa_1] & g_{r-1}[\kappa_{r-1}, \ldots, \kappa_2] \end{pmatrix}$$

$$\begin{pmatrix} g_0[\kappa_r, \ldots, \kappa_3] \\ \vdots \\ g_{r-1}[\kappa_r, \ldots, \kappa_3] \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix} \times$$

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & (\kappa_{r-1} - \kappa_1)^{-1} & \\ & & & & (\kappa_r - \kappa_2)^{-1} \end{pmatrix} B_1 C_1.$$

By repeating this procedure gives the factorization. □

The factorization of D makes it easy to calculate the de-

terminant. First we note that

$$detB_j = 1, \qquad\qquad\qquad j = 1,\ldots,r-1,$$

$$detC_j = \prod_{k=1}^{j}(\kappa_{r-j+k} - \kappa_k)^{-1}, \quad j = 1,\ldots,r-1.$$

Therefore,

$$\begin{aligned}
\prod_{j=1}^{r-1} detC_j &= \prod_{j=1}^{r-1}\prod_{k=1}^{j}(\kappa_{r-j+k}-\kappa_k)^{-1} \\
&= (\kappa_r-\kappa_1)^{-1}(\kappa_{r-1}-\kappa_1)^{-1}\ldots(\kappa_2-\kappa_1)^{-1} \\
&\qquad (\kappa_r-\kappa_2)^{-1}\ldots(\kappa_3-\kappa_2)^{-1}\ldots(\kappa_r-\kappa_{r-1})^{-1} \\
&= \prod_{r\geq j>i\geq 1}(\kappa_j-\kappa_i)^{-1} \\
&= (-1)^{\frac{1}{2}(r-1)r}\prod_{r\geq j>i\geq 1}(\kappa_i-\kappa_j)^{-1} \\
&= (-1)^{\frac{1}{2}(r-1)r}\prod_{r\geq j>i\geq 1}(\kappa_i-\kappa_j)^{-1}.
\end{aligned}$$

Now it only remains to calculate the determinant of $\tilde{D}$. In the <u>outflow</u> case we have with $\gamma_j = (\kappa_j - 1)^q$

$$\tilde{D} = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_r \\ \frac{\gamma_1}{\kappa_1} & \frac{\gamma_2}{\kappa_2} & \cdots & \frac{\gamma_r}{\kappa_r} \\ \vdots & & & \vdots \\ \frac{\gamma_1}{\kappa_1^{r-1}} & \frac{\gamma_2}{\kappa_2^{r-1}} & \cdots & \frac{\gamma_r}{\kappa_r^{r-1}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{\kappa_1} & \frac{1}{\kappa_2} & \cdots & \frac{1}{\kappa_r} \\ \vdots & & & \vdots \\ \frac{1}{\kappa_1^{r-1}} & \frac{1}{\kappa_2^{r-1}} & \cdots & \frac{1}{\kappa_r^{r-1}} \end{pmatrix}$$

$$\begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_r \end{pmatrix} = \mathring{\tilde{D}}G,$$

where the $\mathring{\tilde{D}}$ is a Vandermonde matrix. Let

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{pmatrix},$$

then the determinant of V is $\prod_{n\geq j>i\geq 0}(x_j - x_i)$. With $n = r-1$ and $x_i = 1/\kappa_{i+1}, i = 0,\ldots,r-1$ we have

$$\begin{aligned}
det\mathring{\tilde{D}} &= \prod_{r-1\geq j>i\geq 0}(\tfrac{1}{\kappa_{j+1}} - \tfrac{1}{\kappa_{i+1}}) = \prod_{r\geq j>i\geq 1}(\tfrac{1}{\kappa_j} - \tfrac{1}{\kappa_i}) \\
&= \prod_{r\geq j>i\geq 1}\frac{\kappa_i-\kappa_j}{\kappa_i\kappa_j},
\end{aligned}$$

and

$$det\tilde{D} = \prod_{r\geq j>i\geq 1}\frac{\kappa_i-\kappa_j}{\kappa_i\kappa_j}\prod_{k=1}^{r}(\kappa_k-1)^q.$$

Finally,

$$\begin{aligned}
detD &= (-1)^{\frac{1}{2}(r-1)r}\prod_{j=1}^{r}\frac{1}{\kappa_j^{r-1}}\prod_{k=1}^{r}(\kappa_k-1)^q \\
&= (-1)^{\frac{1}{2}(r-1)r}\prod_{j=1}^{r}\frac{(\kappa_j-1)^q}{\kappa_j^{r-1}}.
\end{aligned}$$

By Lemma 1.2, there is no eigenvalue with $Re\ \tilde{s} > 0$ for the outflow problem.

In the <u>inflow</u> case we have

$$\tilde{D} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{(\kappa_1-1)^2}{\kappa_1} & \frac{(\kappa_2-1)^2}{\kappa_2} & \cdots & \frac{(\kappa_r-1)^2}{\kappa_r} \\ \vdots & & & \vdots \\ \frac{(\kappa_1-1)^{2(r-1)}}{\kappa_1^{r-1}} & \frac{(\kappa_2-1)^{2(r-1)}}{\kappa_2^{r-1}} & \cdots & \frac{(\kappa_r-1)^{2(r-1)}}{\kappa_r^{r-1}} \end{pmatrix}.$$

Since this also is a Vandermonde matrix we immediately get with $x_i = \frac{(\kappa_{i+1}-1)^2}{\kappa_{i+1}}$

$$\begin{aligned}
det\tilde{D} &= \prod_{r\geq j>i\geq 1}(\tfrac{(\kappa_j-1)^2}{\kappa_j} - \tfrac{(\kappa_i-1)^2}{\kappa_i}) \\
&= \prod_{r\geq j>i\geq 1}\frac{(\kappa_j-\kappa_i)(\kappa_i\kappa_j-1)}{\kappa_i\kappa_j}, \quad r > 1.
\end{aligned}$$

Thus,

$$detD = \begin{cases} \prod\limits_{r\geq j>i\geq 1}(1 - \tfrac{1}{\kappa_i\kappa_j}), & r > 1 \\ 1, & r = 1, \end{cases}$$

which only can be zero if $\kappa_i\kappa_j = 1$, $i \neq j$, $1 \leq i,j \leq r$, $r > 1$. By Lemma 1.2 this is not possible for $Re\ \tilde{s} > 0$, and there are no eigenvalue to the inflow problem.

We have proved

Lemma 2.4 *There are no eigenvalues $\tilde{s}$ with $Re\ \tilde{s} > 0$ to the eigenvalue problem (12) with outflow (13) or inflow (14) boundary conditions.*

Finally, we have to show that there are no generalized eigenvalues when $Re\ \tilde{s}$ goes to zero. We have

Lemma 2.5 *There is constant $\delta > 0$ such that, on any compact set $|\tilde{s}| \leq C$, $Re\ \tilde{s} \geq 0$, the roots $\kappa_1, \ldots, \kappa_r$ of the characteristic equation (16) satisfy the inequalities*

$$|\kappa_j - 1| \geq \delta, \ j = 1,\ldots,r \ if \ a > 0,$$

$$|1 - \tfrac{1}{\kappa_i\kappa_j}| \geq \delta, \ i \neq j, \ 1 \leq i,j \leq r \ if \ a < 0.$$

Proof

The roots are continuous functions of $\tilde{s}$. Therefore, the inequalities can only be violated if for some $\tilde{s}$, $Re\ \tilde{s} \geq 0$

$\kappa_j = 1$ when $a > 0$ or $\kappa_i \kappa_j = 1$, $i \neq j$ when $a < 0$.

The first statement of Lemma 1.2 tells us that this cannot happen for $Re\ \tilde{s} > 0$.

Let $a > 0$ and $\kappa_j = 1$, then from (16) $\tilde{s} = 0$. However, the third statement of Lemma 1.2 tells us that $\kappa_j = -1$ and we have a contradiction.

Let $a < 0$ and $\kappa_i \kappa_j = 1$, $i \neq j$, then (16) implies

$$\frac{\tilde{s}}{a} = \sum_{\nu=1}^{r} \alpha_\nu (\kappa_i^\nu - \kappa_i^{-\nu}) = \sum_{\nu=1}^{r} \alpha_\nu (\kappa_j^\nu - \kappa_j^{-\nu})$$

$$= \sum_{\nu=1}^{r} \alpha_\nu (\kappa_i^{-\nu} - \kappa_i^\nu) = -\sum_{\nu=1}^{r} \alpha_\nu (\kappa_i^\nu - \kappa_i^{-\nu}) = -\frac{\tilde{s}}{a}.$$

Thus, $\tilde{s} = 0$ and we have a contradiction since $\kappa_i \kappa_j \neq 1$ according to statement two and three of Lemma 1.2. This proves the lemma. □

2.2 The main results

We now have the main result

Theorem 2.1 *The approximation defined in (4), (7) and (10) is strongly stable and the error of the solution is of order h^{2r} if $q \geq 2r$.*

Proof

From [5] it follows that the approximation is strongly stable since it has no eigenvalues or generalized eigenvalues and the operator is semi-bounded for the Cauchy problem. Therefore it remains only to validate that the error of the solution is h^{2r}. Let u be a smooth function and denote by $e_j(t) = u(x_j, t) - v_j(t)$ the error, then we have

$$\frac{de_j(t)}{dt} = aQe_j(t) + h^{2r}F_j(t), \quad j = 1, 2, \ldots,$$

$$e_j(0) = 0, \qquad\qquad j = 1, 2, \ldots$$

For the boundary conditions we have, if $a < 0$

$$h^{2\nu}a^{2\nu}(D_+D_-)^\nu e_0(t)$$

$$= h^{2\nu}\left(a^{2\nu}(D_+D_-)^\nu u(0,t) - a^{2\nu}(D_+D_-)^\nu v_0(t)\right)$$

$$= h^{2\nu}\left(a^{2\nu}\sum_{j=0}^{\infty} c_j^\nu h^{2j}\frac{\partial^{2(\nu+j)}u(0,t)}{\partial x^{2(\nu+j)}}\right.$$

$$\left. -a^{2\nu}\sum_{j=0}^{r-\nu-1} c_j^\nu h^{2j}\frac{g^{(2(\nu+j))}(t)}{a^{2j}}\right)$$

$$= c_{r-\nu}^\nu h^{2r}\frac{g^{(2r)}(t)}{a^{2(r-\nu)}} + O(h^{2r+2}) = O(h^{2r}),$$

$$\nu = 0, \ldots, r-1.$$

We note that for $\nu = 0$, $c_r^0 = 0$, and therefore the error of the analytic boundary condition $e_0 = 0$.

For $a > 0$,

$$(hD_+)^q e_{-\nu}(t) = (hD_+)^q u(x_{-\nu}, t) - (hD_+)^q v_{-\nu}(t)$$

$$= (hD_+)^q u(x_{-\nu}, t)$$

$$= h^q \frac{\partial^q u(x_{-\nu}, t)}{\partial x^q} + O(h^{q+1}) = O(h^q),$$

$$\nu = 0, 1, \ldots, r-1.$$

Therefore, from the strong stability and since the forcing is of order $O(h^{2r} + h^q)$, we have the following estimate for the error

$$\|e(t)\|_h^2 = const \int_0^t (\|h^{2r}F(\tau)\|_h^2 + |O(h^{2r})|^2)d\tau = O(h^{4r}),$$

if $q \geq 2r$. □

3 Systems

Consider the system

$$(21) \qquad u_t = Au_x, \quad 0 \leq x < \infty, \ t \geq 0,$$

with initial conditions

$$(22) \qquad u(x, 0) = f(x).$$

Approximate by a finite difference approximation of order $2r$

$$(23) \qquad \frac{dv_j(t)}{dt} = Qv_j(t), \quad j = 1, 2, \ldots,$$

$$v_j(0) = f_j, \qquad j = 1, 2, \ldots,$$

where

$$Q = A\frac{1}{h}\sum_{\nu=-r}^{r} \alpha_\nu E^\nu.$$

Since A can be diagonalized, we can assume A having diagonal form with

$$A = \begin{pmatrix} \Lambda^I & \\ & \Lambda^{II} \end{pmatrix}, \ \Lambda^I > 0, \ \Lambda^{II} < 0.$$

The boundary conditions can be written as

$$(24) \qquad u^{II}(0, t) = Ru^I(0, t) + g(t).$$

Differentiation of the boundary conditions (24) and the differential equation (21) give us

$$\frac{\partial^{2\nu}u^{II}(0,t)}{\partial x^{2\nu}} = S_\nu \frac{\partial^{2\nu}u^I(0,t)}{\partial x^{2\nu}} + (\Lambda^{II})^{-2\nu}g^{(2\nu)}(t),$$

$$S_\nu = (\Lambda^{II})^{-2\nu}R(\Lambda^I)^{2\nu}, \ \nu = 0, 1, \ldots.$$

$$(25)$$

As boundary conditions for the in-going characteristic variable we approximate (25) for $\nu = r - 1, \ldots, 0$ by

$$(26) \quad Q_\nu v_0^{II}(t) \;=\; S_\nu Q_\nu v_0^I(t) + (\Lambda^{II})^{-2\nu} g^{(2\nu)}(t),$$

where

$$Q_\nu \;=\; (D_+ D_-)^\nu - \sum_{j=1}^{r-\nu-1} c_j^\nu \, h^{2j} \, Q_{\nu+j}.$$

The reason to define Q_ν this way is that $Q_\nu u(0)$ will be an approximation of $u^{(2\nu)}(0)$ of order $h^{2r-2\nu}$. This is exactly what we need for the boundary conditions to be of order h^{2r}. For the outgoing characteristic variable we use extrapolation conditions

$$(27) \qquad D_+^q v_{-\nu}^I(t) \;=\; 0, \quad \nu = 0, 1, \ldots, r - 1.$$

We have the following result

Theorem 3.1 *The approximation defined by (23), (26) and (27) is strongly stable and the error of the solution is of order 2r if $q \geq 2r$.*

Proof
The v^I approximation is decoupled from v^{II} and is already discussed and strongly stable. We can now think of v^I as a given function and write the boundary conditions for v^{II} as

$$Q_\nu v_0^{II}(t) \;=\; g_{-\nu}(t),$$

$$g_{-\nu}(t) \;=\; S_\nu Q_\nu v_0^I(t) + (\Lambda^{II})^{-2\nu} g^{(2\nu)}(t),$$

$$\nu = 0, \ldots, r - 1.$$

Now we can think of the approximations of v^{II} as consisting of scalar equations which we already have discussed. By theorem 1.1 they are strongly stable.

Let u be a smooth solution. Denote by, $e_j(t) = u(x_j, t) - v_j(t)$, the error, and we obtain the system

$$\frac{de_j(t)}{dt} \;=\; Q e_j(t) + h^{2r} F_j, \quad j = 1, 2, \ldots,$$

$$e_j(0) \;=\; 0, \qquad\qquad j = 1, 2, \ldots.$$

For the inflow part of the boundary conditions we have

$$h^{2\nu} \left(Q_\nu e_0^{II}(t) - S_\nu Q_\nu e_0^I(t) \right)$$

$$= \; h^{2\nu} \left(Q_\nu u^{II}(0, t) - S_\nu Q_\nu u^I(0, t) - (\Lambda^{II})^{-2\nu} g^{(2\nu)}(t) \right)$$

$$= \; h^{2\nu} \left(\frac{\partial^{2\nu} u^{II}(0,t)}{\partial x^{2\nu}} + O(h^{2r-2\nu}) - \right.$$

$$S_\nu \left(\frac{\partial^{2\nu} u^I(0,t)}{\partial x^{2\nu}} + O(h^{2r-2\nu}) \right) - (\Lambda^{II})^{-2\nu} g^{(2\nu)}(t) \right)$$

$$= \; O(h^{2r}), \quad \nu = 0, \ldots, r - 1,$$

and for the outflow part

$$
\begin{aligned}
(hD_+)^q e_{-\nu}^I(t) &= (hD_+)^q u^I(x_{-\nu}, t) - (hD_+)^q v_{-\nu}^I(t) \\
&= (hD_+)^q u^I(x_{-\nu}, t) \\
&= h^q \frac{\partial^q u^I(x_{-\nu}, t)}{\partial x^q} + O(h^{q+1}) \\
&= O(h^q), \quad \nu = 0, 1, \ldots, r - 1.
\end{aligned}
$$

This shows that the forcing is of order $O(h^{2r}) + O(h^q)$ and the desired estimate follows from the strong stability of the approximation.   □

4 Numerical results

Consider the two-dimensional wave equations

$$\mathbf{u}_t = A_1 \mathbf{u}_{x_1} + A_2 \mathbf{u}_{x_2}, \quad 0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 1, \quad t \geq 0,$$
(28)
where $\mathbf{u} = (p \; u \; v)^T$ and

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The components of $\mathbf{u}$ are pressure and the velocity in the x_1 and x_2 direction. With the boundary conditions

$$
\begin{aligned}
(29) \quad & p(x_1, 0, t) = p(x_1, 1, t) = \\
& p(0, x_2, t) = p(1, x_2, t) = 0, \quad t \geq 0,
\end{aligned}
$$

and initial conditions

$$
\begin{aligned}
& p(x_1, x_2, 0) = \sin(\omega_1 x_1) \sin(\omega_2 x_2), \\
(30) \quad & u(x_1, x_2, 0) = 0, \\
& v(x_1, x_2, 0) = 0,
\end{aligned}
$$

where $\omega_1 = m\pi$ and $\omega_2 = n\pi$, $m, n = 1, 2, \ldots$, the exact solution is

$$
\begin{aligned}
& p(x_1, x_2, t) = \sin(\omega_1 x_1) \sin(\omega_2 x_2) \cos(\sqrt{\lambda}\, t), \\
& u(x_1, x_2, t) = -\frac{\omega_1}{\sqrt{\lambda}} \cos(\omega_1 x_1) \sin(\omega_2 x_2) \sin(\sqrt{\lambda}\, t), \\
& v(x_1, x_2, t) = -\frac{\omega_2}{\sqrt{\lambda}} \sin(\omega_1 x_1) \cos(\omega_2 x_2) \sin(\sqrt{\lambda}\, t),
\end{aligned}
$$
(31)
where $\lambda = \omega_1^2 + \omega_2^2$. Let h_1 and h_2 be mesh widths in the x_1- and x_2-directions, and divide the axis into intervals of length h_1 and h_2 respectively. For $i = -r + 1, \ldots, N_{x_1} + r - 1$ and $j = -r + 1, \ldots, N_{x_2} + r - 1$ we use the notation

$$\mathbf{u}_{i,j}^h(t) = \mathbf{u}^h(x_{1i}, x_{2j}, t), \quad x_{1i} = i\, h_1, \quad x_{2j} = j\, h_2,$$

$$h_1 N_{x_1} = h_2 N_{x_2} = 1.$$

We approximate (28) by a centered difference approximation of order $2r$

$$\frac{d\,\mathbf{u}_{i,j}^h}{dt} = (A_1 Q_{x_1} + A_2 Q_{x_2})\,\mathbf{u}_{i,j}^h, \quad i = 1, \ldots, N_{x_1} - 1,$$
$$j = 1, \ldots, N_{x_2} - 1,$$

(32)

where $\mathbf{u}_{i,j}^h = (p^h\ u^h\ v^h)_{i,j}^T$ and

$$Q_{x_-} = \frac{1}{h_-} \sum_{\nu=-r}^{r} a_\nu E_\gamma^\nu, \qquad \gamma = 1, 2,$$
$$E_1 \mathbf{u}_{i,j}^h = \mathbf{u}_{i+1,j}^h, \quad E_2 \mathbf{u}_{i,j}^h = \mathbf{u}_{i,j+1}^h.$$

To get extra boundary conditions for the numerical scheme we differentiate the boundary conditions (29) with respect to time and use the differential equation (28) to obtain

(33) $\quad \frac{\partial^{2\nu} p(j, x_2, t)}{\partial x_1^{2\nu}} = \frac{\partial^{2\nu} p(x_1, j, t)}{\partial x_2^{2\nu}} = 0, \quad j = 0, 1.$

We approximate (33) for $\nu = 0, \ldots, r - 1$ with

$(D_{-,x_1} D_{-,x_1})^{2\nu} p_{i,j}^h = 0, \quad i = 0, N_{x_1}, j = 1, \ldots, N_{x_2} - 1.$

$(D_{-,x_2} D_{-,x_2})^{2\nu} p_{i,j}^h = 0, \quad j = 0, N_{x_2}, i = 1, \ldots, N_{x_1} - 1,$

(34)

where $D_{-,x}$ and $D_{-,x}$ are the usual forward and backward differences in the x-direction. At $x_1 = 0, 1$ we need boundary conditions for u^h and at $x_2 = 0, 1$ for v^h. These conditions are obtained by extrapolation of the locally outgoing characteristic variables. Thus, for $\nu = 0, \ldots, r - 1$ we have

$$(h_1 D_{+,x_1})^{2r}(p^h - u^h)_{-\nu, j} =$$
$$(h_1 D_{-,x_1})^{2r}(p^h + u^h)_{N_{x_1}+\nu, j} = 0, j = 1, \ldots, N_{x_2} - 1,$$
(35)
$$(h_2 D_{+,x_2})^{2r}(p^h - v^h)_{i, -\nu} =$$
$$(h_2 D_{-,x_2})^{2r}(p^h + v^h)_{i, N_{x_2}+\nu} = 0, \quad i = 1, \ldots, N_{x_1} - 1.$$

We now use the numerical boundary conditions (34) and (35) to modify the operator close to the boundary. The reason why we solve the differential equation only at interior points, $i = 1, \ldots, N_{x_1} - 1, j = 1, \ldots, N_{x_2} - 1$, is that it simplifies the implementation, and since $p = 0$ at the boundary, we will have $u = 0$ at $x_2 = 0, 1$ and $v = 0$ at $x_1 = 0, 1$. Furthermore, $u_{i,j}^h$ at $i = 0, N_{x_1}$ and $v_{i,j}^h$ at $j = 0, N_{x_2}$ are given by the extrapolation conditions (35) with $\nu = 0$.

Figure 1 shows the pressure component of the numerical solution obtained using the sixth-order scheme, (32), (34) and (35) with $r = 3$.
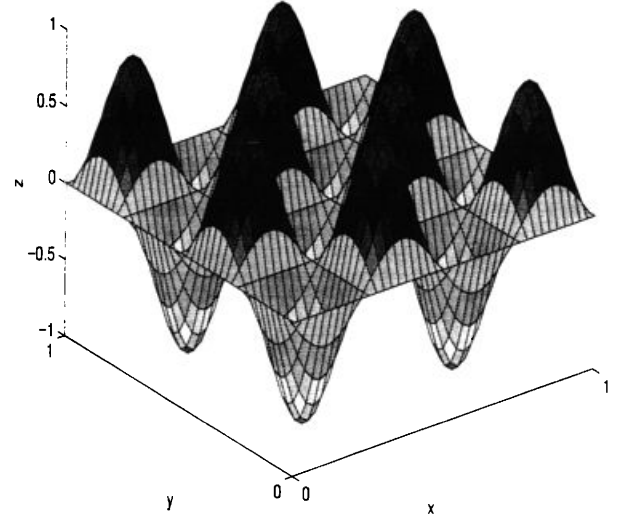

Figure 1: Pressure component of numerical solution of the two-dimensional wave equation at $t = 0.5$. $\omega_1 = \omega_2 = 4\pi$.

4.1 Convergence rate of high-order methods

To analyze the convergence rate of the numerical solution we make a grid refinement study on the two-dimensional wave equation. The Log_{10} of the L_2 error, $\text{Log}_{10}(\|\mathbf{u} - \mathbf{u}^h\|_2)$, is computed at a fixed time $t = T$, and the convergence rate between two grid densities are plotted. For the second- and fourth-order scheme a fourth-order Runge-Kutta scheme was used for time-integration, and for the sixth-order scheme a sixth-order Runge-Kutta. The time step was chosen such that the error of the time-discretization was smaller than the error of the space-discretization. The convergence rate was computed as

$$q = \log_{10}\left(\frac{\|\mathbf{u} - \mathbf{u}^{h_1}\|_2}{\|\mathbf{u} - \mathbf{u}^{h_2}\|_2}\right) / \log_{10}\left(\frac{h_1}{h_2}\right).$$

The results are shown in table 4.1.

The results in table 4.1 agrees well with the theory of Gustafsson [3] and [4] which predicts that boundary conditions of order $p - 1$ must be imposed to retain pth-order global accuracy. Since all boundary conditions considered here are of order $2r - 1$ we get global accuracy of order $2r$.

4.2 Efficiency of high-order methods

The efficiency of high-order methods compared with second order methods has been studied in [8] and [11]. The

	Second-order		Fourth-order		Sixth-order	
Grid	$\text{Log}(L_2)$	q	$\text{Log}(L_2)$	q	$\text{Log}(L_2)$	q
21	-1.176		-2.708		-4.262	
41	-1.753	1.92	-3.956	4.14	-6.622	7.83
81	-2.344	1.96	-5.177	4.06	-8.149	5.07

Table 1: Grid convergence of schemes on two-dimensional wave equation with $\omega_1 = \omega_2 = 2\pi$, $T = 1$.

conclusion is that the high-order methods are more efficient than low-order ones for hyperbolic problems with smooth solutions, except when very low accuracy in the solution is needed.

As a test of the efficiency of the fourth- and sixth-order methods compared with the second-order method, we compute the numerical solution and compare it with the exact solution of the two-dimensional wave equation. This is done at a fixed time, $t = 0.5$, on successively refined grids, with $N_{x_1} = N_{x_2} = N$, and for different frequencies $\omega_1 = \omega_2 = \omega$. On each grid we compute the relative error, $\|\mathbf{u} - \mathbf{u}^h\|_2 / \|\mathbf{u}\|_2$, and measure the consumed CPU time T_{cpu}. For the second-order and the fourth-order schemes a fourth-order Runge-Kutta scheme with four stages was used to integrate in time. For the sixth-order scheme a sixth-order Runge-Kutta with seven stages was used. For all Runge-Kutta schemes the time step was chosen such that the error in the time-discretization was of the same order as the error of the space-discretization and as close to the stability limit as possible. All computations was done on a SUN Spark-Station 10 equipped with a 40 MHz processor and without external cache. The results are presented below.

Table 4.2-4.2 shows clearly the high efficiency of the fourth- and sixth-order methods compared with the second-order one, this is true in particular for high frequencies and high accuracy requirements in the solution. If we want to compute the solution of the problem with $\omega = 4\pi$, with a relative error of 0.001, the second-order method would need approximately 0.9 CPU hours while the fourth-order method would need less than 29 seconds and the sixth-order one less than 15 seconds. For lower frequencies and lower accuracy requirements in the solution the gain is not that big. In table 4.2 a relative error of 0.01 would require approximately 0.5 second CPU time for the second-order method, 0.1 second for the fourth-order method and less than 0.3 seconds for the sixth-order

	Second-order		Fourth-order		Sixth-order	
N	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}
9	$5.88 \cdot 10^{-2}$	0.03	$3.46 \cdot 10^{-3}$	0.09	$5.01 \cdot 10^{-4}$	0.32
17	$1.63 \cdot 10^{-2}$	0.14	$2.43 \cdot 10^{-4}$	0.57	$2.47 \cdot 10^{-6}$	2.0
33	$4.10 \cdot 10^{-3}$	0.90	$1.50 \cdot 10^{-5}$	3.8	$6.62 \cdot 10^{-8}$	15
65	$1.02 \cdot 10^{-3}$	6.7	$8.95 \cdot 10^{-7}$	29	$9.59 \cdot 10^{-10}$	115
129	$2.56 \cdot 10^{-4}$	52	$5.36 \cdot 10^{-8}$	228	$1.22 \cdot 10^{-11}$	911
257	$6.40 \cdot 10^{-5}$	415	$3.26 \cdot 10^{-9}$	1822	$1.52 \cdot 10^{-13}$	7268
513	$1.60 \cdot 10^{-5}$	3305				

Table 2: Relative error and consumed CPU time. $\omega = \pi$

	Second-order		Fourth-order		Sixth-order	
N	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}
9	$5.03 \cdot 10^{-1}$	0.03	$1.01 \cdot 10^{-1}$	0.10	$2.59 \cdot 10^{-2}$	0.33
17	$1.15 \cdot 10^{-1}$	0.14	$7.75 \cdot 10^{-3}$	0.57	$4.50 \cdot 10^{-4}$	2.0
33	$2.72 \cdot 10^{-2}$	0.90	$4.44 \cdot 10^{-4}$	3.8	$7.38 \cdot 10^{-6}$	15
65	$6.74 \cdot 10^{-3}$	6.7	$2.52 \cdot 10^{-5}$	29	$8.66 \cdot 10^{-8}$	115
129	$1.70 \cdot 10^{-3}$	52	$1.48 \cdot 10^{-6}$	228	$9.95 \cdot 10^{-10}$	913
257	$4.29 \cdot 10^{-4}$	415	$8.96 \cdot 10^{-8}$	1818	$1.19 \cdot 10^{-11}$	7277
513	$1.08 \cdot 10^{-4}$	3312				

Table 3: Relative error and consumed CPU time, $\omega = 2\pi$

| | Second-order | | Fourth-order | | Sixth-order | |
N	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}
9	$1.29 \cdot 10^0$	0.03	$9.50 \cdot 10^{-1}$	0.10	$1.66 \cdot 10^0$	0.33
17	$7.84 \cdot 10^{-1}$	0.14	$1.69 \cdot 10^{-1}$	0.58	$4.63 \cdot 10^{-2}$	2.0
33	$2.67 \cdot 10^{-1}$	0.91	$1.09 \cdot 10^{-2}$	3.9	$4.40 \cdot 10^{-4}$	15
65	$6.10 \cdot 10^{-2}$	6.8	$6.55 \cdot 10^{-4}$	29	$2.73 \cdot 10^{-6}$	116
129	$1.45 \cdot 10^{-2}$	52	$4.03 \cdot 10^{-5}$	229	$5.05 \cdot 10^{-8}$	913
257	$3.59 \cdot 10^{-3}$	416	$2.51 \cdot 10^{-6}$	1817	$8.86 \cdot 10^{-10}$	7274
513	$8.95 \cdot 10^{-4}$	3314				

Table 4: Relative error and consumed CPU time, $\omega = 4\pi$

| | Second-order | | Fourth-order | | Sixth-order | |
N	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_2}{\|\mathbf{u}\|_2}$	T_{cpu}
9	$1.00 \cdot 10^0$	0.02	$1.00 \cdot 10^0$	0.10	$1.00 \cdot 10^0$	0.32
17	$1.14 \cdot 10^0$	0.13	$1.41 \cdot 10^0$	0.57	$9.87 \cdot 10^{-1}$	2.0
33	$9.85 \cdot 10^{-1}$	0.90	$2.72 \cdot 10^{-1}$	3.8	$3.84 \cdot 10^{-2}$	15
65	$5.22 \cdot 10^{-1}$	6.7	$1.97 \cdot 10^{-2}$	29	$3.76 \cdot 10^{-4}$	115
129	$1.23 \cdot 10^{-1}$	52	$1.25 \cdot 10^{-3}$	228	$6.33 \cdot 10^{-6}$	911
257	$2.90 \cdot 10^{-2}$	415	$7.84 \cdot 10^{-5}$	1822	$1.09 \cdot 10^{-7}$	7268
513	$7.15 \cdot 10^{-3}$	3315				

Table 5: Relative error and consumed CPU time, $\omega = 8\pi$

one. Thus, it is only for a relative error of the order 0.1 and low frequencies that the second-order method can compete with the high-order ones. Tests with a three-stage second-order Runge-Kutta in combination with the second-order scheme in space was also made. However, the combination second-order in space and fourth-order in time turned out to be more efficient.

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