Divergence Stability of Certain Increasing Order Finite Element Methods for Elliptic and Semi-Elliptic Problems

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Abstract

We introduce and analyze *stable* discrete spaces with quasi-optimal approximation properties (with respect to increasing polynomial degree). This will pertain to some general classes of problems: scalar and systems of elliptic as well as semi-elliptic (Stokes') problems.

Key words: p version, Galerkin spectral element method, divergence stability.

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1 Introduction

High-order mixed methods for solving some classes of diffusion, elasticity and fluid-flow problems lead to some interesting questions on stability and approximability. It is, for some of these problems, possible to achieve high accuracy by using a finite element technique with high-order piecewise-polynomials on a subdivision of the domain. These methods generally go under names such as p or $h-p$ versions of the finite element method or Galerkin spectral element methods.

This note will concentrate on the theory (and practice) of divergence stability.

In view of the lack of stability (inf-sup constant going to zero as the polynomial degree tends to infinity, cf. [18]) for some "natural" choices of discrete spaces – and its effects such as the extent to which the approximation of the velocity/pressure 'locks' – as pertaining to Stokes' (cf. [18] [14] [15] [16] [19] [4]) as well as scalar elliptic problems on a bounded, polygonal, plane domain (cf. [9] [15] [17] [13]), we are interested in the question of whether or not it is possible to define stable discrete spaces.

In §2 we develop some basic constructions, and we then introduce *stable* discrete spaces with quasi-optimal approximation properties (with respect to increasing polynomial degree). This is done for scalar elliptic problems in §3 and, in §4, it is done for semi-elliptic (Stokes') systems of equations.

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2 Basic notation and definitions

Let Ω be a bounded, simply connected domain with either smooth or piecewise curvilinear boundary Γ (with finitely many segments).

Let Sobolev spaces and the norms specifying their topologies, $(H^k(\Omega), \|\cdot\|_k)$ and $(H^s(\Gamma), |\cdot|_s)$, be defined as in [1] or [10]. We identify $H^0(\Omega)$ with $L^2(\Omega)$ and the L^2-inner product is denoted $(\cdot,\cdot)$. (We will, when convenient and hopefully without confusion, at times use the latter to also denote an ordered pair.) Let

$$(1) \quad H(\text{div}, \Omega) \stackrel{\text{def}}{=} ([C^\infty(\Omega)]^2)^{\text{closure under } \|\cdot\|_{H(\text{div})}},$$

where we take the closure with respect to the norm defined by

$$\|\chi\|^2_{H(\text{div})} \stackrel{\text{def}}{=} \|\chi\|_0^2 + \|\nabla\cdot\chi\|_0^2.$$

Then we select (but not yet explicitely) two subspaces:

$$(2) \qquad X(\Omega) \subseteq H(\text{div}, \Omega), \text{ and}$$

$$(3) \qquad Y(\Omega) \subseteq L^2(\Omega),$$

(which are again to be Hilbert spaces). When it is clear from the context we will use X and Y to denote $X(\Omega)$ and $Y(\Omega)$.

We may then define the divergence operator, *div*, on X:

$$(4) \qquad \text{div} : X \ni v \mapsto \nabla\cdot v \in Y; \quad \text{div} \in \mathcal{B}(X, Y),$$

through the classical definition $(v_{1,x} + v_{2,y})$ and density of $C^\infty(\Omega)$ in $L^2(\Omega)$. We associate a bilinear form to the divergence operator:

Definition 2.1 The bilinear form

$$b : X \times Y \ni (v, q) \mapsto b(v, q) \in \mathbb{R}$$

is given by

$$b(v, q) = -\int_\Omega \nabla \cdot v \, q \, dx. \tag{5}$$

□

Similar to (4), we have – for completeness – the operators *curl* and *grad*, classically defined as

$$\nabla \times \phi = (-\phi_y, \phi_x) = (\nabla\phi)^\perp \text{ with } \nabla\phi = (\phi_x, \phi_y),$$

extendable to $\mathcal{B}(H^1, H(\mathrm{div}))$ and $\mathcal{B}(H^1, [L^2]^2)$, respectively.

Using Galerkin mixed methods we will seek weak solutions of elliptic or semi-elliptic problems in two sequences of closed subspaces $X_N \subseteq X$ and $Y_N \subseteq Y$. The index N may be used as an indication of the dimension of the subspaces which is most often a function of some discretization parameters (such as mesh size, h, or degree of polynomials, k, p, or r).

We assume that the variational formulation of the elliptic or semi-elliptic problem involves a separate bilinear form a.

Definition 2.2 Let there, in addition, be given a bilinear form

$$a : X \times X \ni (u, v) \mapsto a(u, v) \in \mathbb{R}.$$

Then, we define the class of *variational problems* under consideration to be of the following saddle-point-type:

$$\begin{array}{rcl} \text{Find } u_N \in X_N & \text{and} & \pi_N \in Y_N \text{ s.t.} \\ a(u_N, v) + b(v, \pi_N) & = & f^1(v) \;\; \forall v \in X_N, \\ b(u_N, q) & = & f^2(q) \;\; \forall q \in Y_N. \end{array} \tag{6}$$

where $f^1 \in X^*$ and $f^2 \in Y^*$ (the dual spaces of X and Y, respectively) are given. □

The fact that the problem (6) is well-posed remains to be verified in the particular cases and will rely on the general framework in [2] and [6]. (Note that we have not excluded the possibility of setting $(X_N, Y_N) = (X, Y)$.) Here we merely consider the case when the problem (6) is *semi-simply set*:

Definition 2.3 The variational problem (6) is said to be *semi-simply set* if, in addition, a is bounded and coercive (over X) and b is bounded (over (X, Y)), i.e. $\exists\ c, C > 0$ such that

$$\begin{array}{rcl} |a(u, v)| & \leq & C\|u\|_X \|v\|_X \quad \forall u, v \in X, \\ \pm a(v, v) & \geq & c\|v\|_X^2 \quad \forall v \in X, \text{ and} \\ |b(v, q)| & \leq & C\|v\|_X \|q\|_Y \forall v \in X, q \in Y. \end{array} \tag{7}$$

where either the + or the − is used uniformly over X. □

If the problem is semi-simply set, we may concentrate on the second inf-sup condition of Brezzi's in order to establish well-posedness and stability. Towards that end, we also merely consider the case when the family of subspaces $\{(X_N, Y_N)\}_N$ conform to the continuous problem in a certain sense:

Definition 2.4 The sequence of pairs of subspaces $\{(X_N, Y_N)\}_N$ is called *Hodge-conforming* in (X, Y) if

1. $X_N \subseteq X$ and $Y_N \subseteq Y$, as well as
2. $\nabla \cdot X_N \subseteq Y_N$.

□

Let us define the affine manifolds (depending on f^2):

$$\begin{array}{rcl} M_N & \stackrel{\text{def}}{=} & \{w \in X_N : b(w, q) = f^2(q),\ \forall q \in Y_N\}, \\ M_0 & \stackrel{\text{def}}{=} & \{w \in X_N : b(w, q) = 0\ , \forall q \in Y_N\}, \text{ and} \\ M & \stackrel{\text{def}}{=} & \{w \in X : b(w, q) = f^2(q),\ \forall q \in Y\}. \end{array} \tag{8}$$

Then, we may reformulate part of our variational problem (6) as:

$$\begin{array}{rcl} \text{Find } u_N \in M_N \text{ s.t.} & & \\ a(u_N, v) & = & f^1(v) \;\; \forall v \in M_0. \end{array} \tag{9}$$

which is useful in certain situations.

Lemma 2.1 (à la Brenner & Scott) *Suppose the variational problem (6) is semi-simply set and the sequence of pairs of subspaces $\{(X_N, Y_N)\}_N$ is Hodge-conforming in (X, Y). Then the following error estimate holds*

$$\|u - u_N\|_X \leq C \left\{ \inf_{v \in M_N} \|u - v\|_X + \inf_{q \in Y_N} \|\pi - q\|_Y \right\}.$$

Proof One merely, but carefully, checks that the arguments from Lemma 8.1.1 in [5] carry over to get

$$c\|u - u_N\|_X \leq \inf_{v \in M_N} \|u - v\|_X + \sup_{w \in M_0 \setminus \{0\}} \frac{a(u - u_N, w)}{\|w\|_X},$$

and then one replaces $a(u, w)$ and $a(u_N, w)$ using (6) and (9) as well as employs that $w \in M_0$ to get the claim. □

In situations where f^2 is particularly simple (0!, 1, or approximated a priori), we get the following lemma.

Lemma 2.2 (à la Scott & Vogelius) *Suppose that the variational problem (6) is semi-simply set, $\exists f_N \in Y_N$ so that $f^2 : Y \ni q \mapsto f^2(q) = (f_N, q) \in \mathbb{R}$, and that the sequence of pairs of subspaces $\{(X_N, Y_N)\}_N$ is Hodge-conforming in (X, Y). Then, $M_N \subset M$ and the following error estimate holds*

$$(10) \qquad \|u - u_N\|_X \le C \left\{ \inf_{v \in M_N} \|u - v\|_X \right\}.$$

Proof by Ceá's lemma. □

In order to obtain quasi-optimal error estimates, it would be very convenient if we could establish:

$$(11) \qquad \inf_{v \in M_N} \|u - v\|_X \le C \left\{ \inf_{w \in X_N} \|u - w\|_X \right\}.$$

As is well-known from [22], this is in the Hodge-conforming case closely related to the concept of divergence-stability (in turn, intimately connected to the second inf-sup condition), which we generalize slightly for our benefit.

Definition 2.5 A family of closed subspaces $\{W_N\}_N \subseteq 2^X$ is called *divergence-stable* with respect to (X, Y) if

1. the spaces $\nabla \cdot W_N$ are closed in Y, and
2. $\exists c > 0$, independent of N, such that

$$(12) \qquad \sup_{w \in W_N} \frac{b(w,q)}{\|w\|_X} \ge c\|q\|_Y, \forall q \in \nabla \cdot W_N;$$

cf. [22]. □

Lemma 2.3 *Suppose (6) is semi-simply set and $\{X_N\}_N$ is divergence-stable with respect to (X, Y). Then (6) is well-posed on $(X_N, \nabla \cdot X_N)$ and (u_N, π_N) is uniformly stable in (X, Y). In addition, the following error estimate holds*

$$\|u - u_N\|_X + \|\pi - \pi_N\|_Y \le C\{\inf_{v \in X_N} \|u - v\|_X + \inf_{q \in Y_N} \|\pi - q\|_Y\}.$$

Proof à la Brezzi or Babuška. □

Proposition 2.1 (à la Scott & Vogelius) *Let the assumptions of Lemma 2.2 be fulfilled. Then the spaces M_N and X_N satisfy the estimate (11) for arbitrary $u \in M$, with a constant C that is independent of u and N if, and only if, $\{X_N\}_N$ is divergence-stable with respect to (X, Y).*

Proof as in [22]. □

As we know from [22], this is equivalent to the existence of a sequence of uniformly good vib'es, i.e., right-inverses to the divergence operators:

$$(13) \quad \begin{aligned} &\mathrm{vib} : Y \ni q \mapsto v \in X, \quad \nabla \cdot v = q; \quad \mathrm{vib} \in \mathcal{B}(Y, X); \\ &\mathrm{vib}_N : Y_N \ni q \mapsto v \in X_N, \quad \mathrm{div}(\mathrm{vib}_N q) = q \quad \forall q \in Y_N, \end{aligned}$$

with a uniform bound $\|\mathrm{vib}_N\|_{\mathcal{B}(Y,X)} \le C$ for C independent of N. (We used, implicitly, the fact that $\{(X_N, Y_N)\}_N$ is Hodge-conforming in (X, Y) to see that $\mathrm{vib}_N \in \mathcal{B}(Y_N, X_N)$.) We are therefore interested in deriving norm estimates for $\mathrm{vib}_N = (\nabla \cdot)^{-1}|_{Y_N}$ in the topology of $\mathcal{B}(Y, X)$.

We will try to create analogues of the well-known Helmholtz decomposition in the plane.

Theorem 2.1 (Helmholtz)
Every function v of $[L^2(\Omega)]^2$ has the following orthogonal decomposition:

$$(14) \qquad v = \nabla q + \nabla \times \phi,$$

where $q \in H^1/\mathbb{R}$ is the only solution of

$$(15) \qquad (\nabla q, \nabla \mu) = (v, \nabla \mu) \quad \forall \mu \in H^1,$$

and $\phi \in H^1_0$ is the only solution of

$$(16) \quad (\nabla \times \phi, \nabla \times \chi) = (v - \nabla q, \nabla \times \chi) \quad \forall \chi \in H^1_0.$$

A proof of the result in this form is given in [10] Thm. 3.2.

Given X, let Φ be a vector space of stream- or (Airy) stress functions (read: pre-curls), i.e. $\nabla \times \Phi \subseteq X$, and Ψ be a vector space of potential functions (read: pre-gradients), i.e. $\nabla \Psi \subseteq X$. Let there be given a sequence of pairs of parental spaces $\Phi_N \subseteq \Phi$ and $\Psi_N \subseteq \Psi$.

Definition 2.6 The pair of spaces (Φ, Ψ), the sequence of pairs of subspaces $\{(\Phi_N, \Psi_N)\}_N$, along with the sequence of subspaces $\{X_N\}_N$ is called *Helmholtz-conforming* in (X, Y) if

1. $X = \nabla \Psi + \nabla \times \Phi$, and
2. $\Phi_N \subseteq \Phi$ and $\Psi_N \subseteq \Psi$, as well as
3. $X_N \subseteq \nabla \times \Phi_N + \nabla \Psi_N$.

□

Next, consider function spaces that are (possibly piecewise) polynomials, (sectionally defined on subsets $\Omega_i \subseteq \Omega$ that are triangles, parallelograms, or at times such with one curved side (coinciding with a part of Γ)).

Definition 2.7 Let

$$\begin{aligned}\mathcal{P}^p &= \text{span}\{x^l\, y^m \ : 0 \le l, m \text{ and } l+m \le p\} \text{ and}\\ \mathcal{Q}^p &= \text{span}\{x^l\, y^m \ : 0 \le l, m \le p\}\end{aligned}$$

be polynomial spaces of *total* and *separate* degree at most p, respectively. □

Definition 2.8 We call Ω an *algebraically simple* domain if $\Gamma = \cup_{j=1}^{J} \Gamma_j$, where $J < \infty$ and each Γ_j is a segment of an algebraic curve in the sense that $\exists p_0$ such that

1. $\overline{\Omega} \cap \{(x,y) \in \mathbb{R}^2 : p_0(x,y) = 0\} = \Gamma$,
2. p_0 is merely a product of at most J polynomials, each irreducible over $\mathbb{R}$, so that, defining
   $\underline{n}_\Omega \stackrel{\text{def}}{=}$ the *separate* $\deg(p_0)$, and
   $\overline{n}_\Omega \stackrel{\text{def}}{=}$ the *total* $\deg(p_0)$,
   each $\deg(p_0)$ is minimal.

(As an example, let $p_0(x,y) = (1-x^2)(1-y^2)$ for $\Omega = S = (-1,1)^2$ with $\underline{n}_S = 2$ and $\overline{n}_S = 4$.) □

Definition 2.9 Let P_N denote the L^2-projection onto Y_N:

$$P_N : Y \ni q \mapsto P_N q \in Y_N, \ (P_N q, s) = (q, s) \ \ \forall s \in Y_N.$$

□

3 Poisson's equation

Let U satisfy the following Poisson problem

$$\begin{aligned} -\Delta U &= f \ \text{ in } \ \Omega\\ U &= g \ \text{ on } \ \Gamma_0 \text{ and}\\ \tfrac{\partial U}{\partial n} &= h \ \text{ on } \ \Gamma_1\end{aligned} \tag{17}$$

$\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. We assume that we may use linearity (or superposition) to subtract off a special function so that we may take vanishing Neumann-data: $h = 0$.

Let $X = \{v \in H(\text{div}) : v \cdot n = 0 \text{ on } \Gamma_1\}$ and $Y = L^2$. Suppose $U \in Y$ and let $u = \nabla U \in X$ so that div $u = \Delta u \in Y$. As b was defined before, we define a and f^i:

$$\begin{aligned} a(u,v) &= -(u,v) \ \ \forall u, v \in X,\\ f^1(v) &= \textstyle\int_{\Gamma_0} g\,(v \cdot n)\, ds \ \ \forall v \in X, \text{ and}\\ f^2(q) &= (f, q) \ \ \forall q \in Y.\end{aligned} \tag{18}$$

Then, a variational formulation of (17) is given by the system (6). This is semi-simply set provided $f \in Y, g \in H^{1/2}(\Gamma_0)$, which is henceforth assumed. As (X,Y) is Hodge-conforming in (X,Y) and X is divergence-stable with respect to (X,Y), the continuous problem is well-posed.

We now construct a new mixed method projection Λ_N with some of the same properties as the Raviart-Thomas projection but with better error estimates (with respect to p). Note that $u = \nabla U$ and $\pi = U$ throughout our discussion of Poisson's equation.

3.1 One element Galerkin mixed method

We will take – as a precursor to the next subsection – the instance of one element: let Ω be a triangle, a parallelogram, or – generally – let the domain be *algebraically simple* and convex. Also, to further simplify, let $g = 0$ and $\Gamma_1 = \emptyset$.

Let $\Psi_N \subseteq p_0 \mathcal{Q}^{p+1-\underline{n}_\Omega} = \mathcal{Q}^{p+1} \cap H_0^1(\Omega)$ and define the discrete spaces $Y_N = \Delta \Psi_N$ and $X_N = \nabla \Psi_N$. Clearly

$$\begin{tikzcd} \Psi_N \arrow[rr, leftrightarrow, "\Delta", "\text{bijection}"'] \arrow[dr, "\nabla"'] & & Y_N \\ & X_N \arrow[ur, "\nabla\cdot"'] & \end{tikzcd}$$

commutes.

Lemma 3.1 *$\{(X_N, Y_N)\}_N$ is Hodge-conforming in (X,Y).*

Proof by construction. □

Lemma 3.2 *$\{X_N\}_N$ is divergence-stable with respect to (X,Y).*

Proof A simple consequence of the bijection property of Δ and the elliptic (merely energy) estimate $\|\nabla(\Delta)^{-1} q\|_X \le C\|q\|_Y$. □

Thus the discrete problem is well-posed. As div $u = \Delta U \in Y$ by assumption, we may define V to be the unique solution to:

$$\begin{aligned} \Delta V &= P_N \Delta U \text{ in } \Omega\\ V &= 0 \text{ on } \Gamma.\end{aligned} \tag{19}$$

Then define

$$\Lambda_N u \stackrel{\text{def}}{=} \nabla V. \tag{20}$$

We collect a few simple properties of the projection Λ_N:

Lemma 3.3 *Let $\Lambda_N : X \to X_N$ be as defined in equation (20). Then Λ_N satisfies the crucial commutative property:*

$$\text{div}\ \Lambda_N u = P_N \text{div}\ u, \quad \forall u \in H(\text{div}),$$

stability in H^1:

$$\|\Lambda_N u\|_1 \leq C\|u\|_1,$$

as well as the quasi-optimal error estimates, in case Ω is a triangle or a parallelogram:

$$\|u - \Lambda_N u\|_s \leq C p^{-r+s-1}\|\text{div } u\|_r, \textit{ for } s = 0, 1.$$

Here, if we wish, we may estimate $\|\text{div } u\|_r \leq \|u\|_{r+1}$. We note the quasi-optimal L^2 estimate which improves upon the estimate for the Raviart-Thomas projection in [20].

Proof The commutative property is seen by inspection. Recall from (20) that $\Lambda_N u = \nabla V$. By elliptic estimates, we have the shift inequalities:

$$\|\nabla V\|_s \leq \|V\|_{s+1} \leq C\|\Delta V\|_{s-1} \quad \text{for } s = 0, 1,$$

so that stability in H^1 is a consequence of:

$$\|\nabla V\|_1 \leq C\|P_N \Delta U\|_0 \leq C\|\Delta U\|_0 \leq C\|u\|_1.$$

For the error estimates, recall also that $u = \nabla U$ and observe that

$$\|\nabla U - \nabla V\|_1 \leq \|U - V\|_2 \leq C\|\text{div } u - P_N \text{div } u\|_0,$$

and use the L^2 estimate in the next lemma. Similarly,

$$\|\nabla U - \nabla V\|_0 \leq \|U - V\|_1 \leq C\|\text{div } u - P_N \text{div } u\|_{-1}$$

and with another application of Lemma 3.4, the claim has been proved. □

Lemma 3.4 *The following quasi-optimal estimates hold:*

$$\|v - P_N v\|_{-s} \leq C p^{-r-s}\|v\|_r, \textit{ for } s = 0, 1$$

for $r \leq 2\alpha_{\min}$ where $\alpha_{\min}$ is π divided by the largest interior angle of any corner of Ω.

Proof Let $s = 0$, and note that, with $\Delta\psi = v$ and $\Delta\psi_N = v_N$ for some $\psi \in \Psi$ and $\psi_N \in \Psi_N$, $\|v - P_N v\|_0 = \|\Delta(\psi - \psi_N)\|_0 \leq C\|\psi - \psi_N\|_2$. This may be bounded from above by $Cp^{-r}\|\psi\|_{r+2} \leq Cp^{-r}\|v\|_r$, provided each of these norms are finite, using approximation results established in [3] on either a standard triangle or a square. Given $v \in H^r$, we may write the solution ψ as a finite sum: $\psi = \sum_i c_i\psi_i + \psi_R$, with $\|\psi_R\|_{r+2} + \sum_i |c_i| \leq C\|v\|_r$ and $\psi_i = \rho^\alpha \chi(\rho) \sum_{j=0}^1 |\log \rho|^j \phi_j(\theta)$, in local polar coordinates (ρ, θ) near a corner of Ω; χ and ϕ_j are smooth with χ vanishing outside a neighborhood of the corner. If the interior angle of the corner is ω, then α is a multiple of π/ω. Now, also by approximation results in [3], $\exists \hat{\psi}_N, \tilde{\psi}_N \in \Psi_N$: $\|\psi_i - \hat{\psi}_N\|_2 \leq Cp^{-2\alpha}$ and $\|\psi_R - \tilde{\psi}_N\|_2 \leq Cp^{-r}\|\psi_R\|_{r+2}$, so that $\|\psi - \psi_N\|_2 \leq Cp^{-r}\|v\|_r$, for $r \leq 2\alpha_{\min}$, the latter being at least four.

For $s = 1$, duality and the projection property yields:

$$\begin{aligned}
\|v - P_N v\|_{-1} &= \sup_{w \in H_0^1} \frac{(v - P_N v, w)}{\|w\|_1} \\
&= \sup_{w \in H_0^1} \inf_{w_N \in Y_N} \frac{(v - P_N v, w - w_N)}{\|w\|_1} \\
&\leq \sup_{w \in H_0^1} \inf_{w_N \in Y_N} \frac{\|v - P_N v\|_0 \|w - w_N\|_0}{\|w\|_1} \\
&\leq Cp^{-1}\|v - P_N v\|_0
\end{aligned}$$

once more employing the L^2 approximation result. □

Remark 3.1 *We sketch a proof of the preceding lemma allowing for r arbitrarily large: redefine Ψ_N by first embedding $\Omega \subset C \subset S$ in a circle C and further in a square S – using the Stein extension [23] – on which we let Ψ_N be defined over S, but solve the Poisson problem for ψ on C and then restrict functions to Ω, see also [9]. One would use approximation results for S, but regularity results for C.*

We note that the collection $\{(\Phi_N, \Psi_N)\}_N$, $\{(\Phi, \Psi)\}$, and $\{X_N\}_N\{(\Phi, \Psi)\}$ is Helmholtz-conforming in (X, Y) with the choices $\Phi_N = 0$, $\Phi = H_0^1$, and $\Psi = H^1/\mathbb{R}$.

Remark 3.2 *We can sketch a proof of the preceding lemma for Ω algebraically simple and convex: redefine Ψ_N by first embedding Ω in a square S on which we perform the preceding constructions and then restrict functions to Ω, see also [9].*

Proposition 3.1 *For this mixed method the following error estimates hold:*

$$\begin{aligned}
\|u - u_N\|_X &\leq Cp^{-r}\|\text{div } u\|_r, \textit{ and} \\
\|U - U_N\|_Y &\leq Cp^{-r}(\|\text{div } u\|_r + \|U\|_r).
\end{aligned}$$

Moreover,

$$\|U - U_N\|_Y \leq Cp^{-r}\|U\|_r.$$

Proof The first two inequalities follow from the Lemmas in this subsection coupled with Lemmas 2.2 and 2.3. The last inequality is a consequence of the analysis in [8]; we note, in particular, that hypotheses (H1)-(H3) and (H5) hold. In addition, (H7) holds with $\Sigma_N = P_N$. Theorem 3 and the estimates on page 275 in [8] then yield the claimed error estimate. □

A curved side of one element coinciding with Γ is proposed to be taken care of as described in [9].

If coupled with an appropriate method of quadrature, this could be used as a spectral method.

3.2 Multiple elements

Let Ω be a convex, polygonal domain (possibly with curvilinear segments of the boundary Γ). Geometrically decompose $\Omega = \cup_{i=1}^{M}\Omega_i$ into triangles or parallelograms in such a way that a pair of distinct Ω_i intersect only in three possible ways: (1) $\emptyset$, (2) a common side, or (3) a common vertex. Let $R = (-1,1)^2$ and $T = \{(x,y) : |x| < 1, \ -1 < y < x\}$ denote a reference square and triangle, respectively. Let F_i be an affine, orientation preserving (i.e. the Jacobian $\det(DF_i) > 0$) mapping which maps Ω_i onto R if Ω_i is a parallelogram and onto T if Ω_i is a triangle.

Then we define the space of piecewise polynomials

$$S^p = \{u \in L^2(\Omega) : \text{ for } 1 \le i \le M,$$
$$(21) \qquad u|_{\Omega_i} \circ (F_i)^{-1} \in \left\{ \begin{array}{l} \mathcal{Q}^p(R) \text{ if } F_i(\Omega_i) = R \\ \mathcal{P}^p(T) \text{ if } F_i(\Omega_i) = T \end{array} \right\}\},$$

and we choose

$$(22) \qquad \Psi_N = S^{p+1} \cap H^2(\Omega) \cap H_0^1(\Omega),$$
$$X_N = \nabla\Psi_N, \quad Y_N = \nabla \cdot X_N = \Delta\Psi_N.$$

In the second to last identity, we understand div as defined on $H(\text{div})$. Thus $\Psi_N \subseteq C^1(\overline{\Omega})$ and $X_N \subseteq [C^0(\overline{\Omega})]^2$, see [7]. The functions in Y_N are allowed to be discontinuous.

Remark 3.3 *We are obviously overshooting with C^1 elements – yielding C^0 ones for X_N – when it would have sufficed to have continuity of the normal components of functions in X_N across inter-element boundaries, i.e. ψ_n continuous across $\partial\Omega_i \cap \partial\Omega_j$. We know that it is possible to define a space Ψ_N achieving this (algebraic conditions!) with quasi-optimal approximation properties – after all, the present C^1 elements would be embedded.*

Definition 3.1 The mixed projection is (extended to be) defined as in (20):

$$\Lambda_N u \stackrel{\text{def}}{=} \nabla V,$$

where we may define V to be the unique solution in $H_0^1(\Omega)$ to:

$$(23) \ - (\nabla V, \nabla W) = (P_N \Delta U, W) \text{ for all } W \in H_0^1(\Omega).$$

as div $u = \Delta U \in Y$ by assumption. Note that $V \in \Psi_N$. □

Instead of going through the lemmas from the previous section one by one, we state the main result.

Proposition 3.2 *This mixed method is well-posed and the following error estimates hold:*

$$\begin{aligned} \|u - u_N\|_X &\le Cp^{-r}\|\text{div } u\|_r, \text{ and} \\ \|U - U_N\|_Y &\le Cp^{-r}(\|\text{div } u\|_r + \|U\|_r). \end{aligned}$$

Proof Lemmas 3.1 and 3.2 hold as before for the new (X_N, Y_N). Lemmas 3.3 and 3.4 hold – modulo an issue on regularity which is addressed next – as before. The problem (23) retains the regularity properties used in Lemmas 3.3 and 3.4 due to Thm. 2.4.3 in [12] (for H^2 regularity) as well as [11] (for higher regularity than H^2). Finally, one again uses Lemma 2.3. □

Increasing degree finite elements of higher degree of continuity have been considered in [29], [28], [27], [21], and [24] - among others.

4 Stokes' equations

Linearized, incompressible, and viscous flows are often modelled by the following Stokes problem in the velocity $(\vec{U})$ – pressure (P) formulation with unit kinematic viscosity:

$$(24) \qquad \begin{aligned} -\Delta\vec{U} + \nabla P &= \vec{F} \quad \text{in} \quad \Omega, \\ \nabla \cdot \vec{U} &= 0 \quad \text{in} \quad \Omega \end{aligned}$$

along with some appropriate boundary conditions (no-slip or stress-free, e.g.) on Γ.

Let $X = [H_0^1]^2$ and $Y = L_0^2 = \{q \in L^2 : (q,1) = 0\}$ for no-slip boundary conditions. Let rigid body motions be denoted $\mathcal{R} = \{v \in [H^1]^2 : \epsilon_{ij}(v) = 0\}$ where $\epsilon_{ij}(v) = (v_{i,j} + v_{j,i})/2$. Then we may reflect stress-free boundary conditions by selecting $\tilde{X} = \mathcal{R}^\perp$ (the orthogonal complement of $\mathcal{R}$ in $[H^1]^2$) and $\tilde{Y} = L^2$. As b was defined before, we define a and f^i:

$$(25) \qquad \begin{aligned} a(u,v) &= (\nabla u, \nabla v) \quad \forall u, v \in X, \\ f^1(v) &= (F, v) \quad \forall v \in X, \text{ and} \\ f^2(q) &= 0 \quad \forall q \in Y. \end{aligned}$$

Then, a variational formulation of (24) is given by the system (6). This is semi-simply set provided $F \in X^*$ which is henceforth assumed. As (X, Y) is Hodge-conforming in (X, Y) and X is divergence-stable with respect to (X, Y), the continuous problem is well-posed. The similar statement for $(\tilde{X}, \tilde{Y})$ also holds, cf. §3-4 in [22] and [26]-[25], provided the compatibility condition $(F, r) = 0$, $\forall r \in \mathcal{R}$ is satisfied. Note that $u = \vec{U}$ and $\pi = P$ throughout our discussion of Stokes' problem. Let us, finally, define a special class of problems (pressures):

$$(26) \qquad \mathbb{P}(\Omega) \stackrel{\text{def}}{=} \{p \in Y : \exists\psi \in H_0^2(\Omega) : p = \Delta\psi\}.$$

4.1 One element Galerkin mixed method

Let Ω be a triangle, a parallelogram, or – modulo approximation properties of underlying polynomial spaces – let the domain be *algebraically simple* and convex.

First the case of no-slip b.c. Let $\Phi_N = \Psi_N = B^p = p_0^2\ \mathcal{Q}^{p+1-2\underline{n}_\Omega}$ where p_0 and $\underline{n}_\Omega$ are defined as in section 2. We now set

$$X_N = \nabla \times B^p \oplus \nabla B^p, \tag{27}$$

and

$$Y_N = \Delta(B^p) = \nabla \cdot X_N. \tag{28}$$

Now

$$\begin{tikzcd} \Phi_N \arrow[rr, "0"] \arrow[dr, "\nabla\times"'] & & 0 \\ & X_N \arrow[ur, "\nabla\cdot"'] & \end{tikzcd} \qquad \begin{tikzcd} \Psi_N \arrow[rr, leftrightarrow, "\Delta", "\text{bijection}"'] \arrow[dr, "\nabla"'] & & Y_N \\ & X_N \arrow[ur, "\nabla\cdot"'] & \end{tikzcd}$$

commute. Essentially, $\nabla \times \Phi_N$ is used for velocity approximation and $\Delta\Psi_N$ for pressure approximation. Now, as in the previous discussions, the isomorphism $\Delta : \Psi_N \longrightarrow Y_N$ can be used to get

$$\|\mathrm{vib}_N\|_{\mathcal{B}(Y;X)} \leq C \tag{29}$$

uniformly. Note the new definitions of X and Y (as compared to the situation in Section 3) which might have made this task much harder, cf. [18] as compared to [9], however now turns out not to be. Lemmas 3.1 and 3.2 hold as before for the new (X_N, Y_N):

Lemma 4.1 *$\{(X_N, Y_N)\}_N$ is Hodge-conforming in (X, Y). Furthermore, $\{X_N\}_N$ is divergence-stable with respect to (X, Y).*

Proof by construction. □

Lemma 3.4 for the new Y_N also hold but merely for $P \in \mathbb{P}(\Omega)$ as we are dealing with p_0^2 to handle no-slip b.c.

Proposition 4.1 *This mixed method is well-posed and the following error estimates hold:*

$$\begin{aligned} \|u - u_N\|_X &\leq Cp^{-r}\|u\|_{r+1}, \text{ and} \\ \|P - P_N\|_Y &\leq Cp^{-r}(\|u\|_{r+1} + \|P\|_r), \end{aligned}$$

provided $P \in \mathbb{P}(\Omega)$.

Proof As the spaces are Hodge-conforming and $M_N \subset M$, we note that $u = \nabla \times \psi$ for some $\psi \in H_0^2(\Omega)$ and that we may approximate this stream function at optimal rates within B^p using results from [24] and [14]. One, in addition to the previously established facts, uses Lemma 2.3. □

Thus we can create p-stable Stokes elements which possess quasi-optimal approximation properties; furthermore the exact solution is solenoidal and – of course – satisfies no-slip boundary conditions. The cost of this was the (old remedy of an) enlargement of the velocity subspace. Note that we have some additional freedom in the choice of what to put in the argument of curl $(\cdot)$ in Definition 2.6. One may also use $\cdot = p_0^2\mathcal{P}^{p+1-2\underline{n}_\Omega}$ with optimal approximation properties, e.g.

We note that the collection $\{(\Phi_N, \Psi_N)\}_N$, $\{(\Phi, \Psi)\}$, and $\{X_N\}_N$ is Helmholtz-conforming in (X, Y) with the choices $\Phi_N = \Psi_N$ as selected, $\Phi = H_0^1$, and $\Psi = H^1/\mathbb{R}$.

For stress-free b.c. we may reduce the exponent of p_0 in the definition of Φ_N and Ψ_N leading to:

Corollary 4.1 *Let $\tilde{\Phi}_N = p_0\ \mathcal{Q}^{p+1-\underline{n}_\Omega} \cap L_0^2$ and $\tilde{\Psi}_N = p_0\ \mathcal{Q}^{p+1-\underline{n}_\Omega}$. Then, this mixed method is well-posed and the following error estimates hold:*

$$\begin{aligned} \|u - u_N\|_X &\leq Cp^{-r}\|u\|_{r+1}, \text{ and} \\ \|P - P_N\|_Y &\leq Cp^{-r}(\|u\|_{r+1} + \|P\|_r). \end{aligned}$$

Proof Please note that $\tilde{X}_N = \nabla \times \tilde{\Phi}_N + \nabla\tilde{\Psi}_N \subset \mathcal{R}^\perp$ as $\mathrm{div} v = 0$, $\vec{\mathrm{curl}}\ v$ is constant in Ω for all $v \in \mathcal{R}$, and $\phi = \psi = 0$ on Γ for all $\phi \in \tilde{\Phi}_N$ and all $\psi \in \tilde{\Psi}_N$. □

The analysis presented here could easily be extended to the case that homogeneous Dirichlet data is given on a part of the boundary, not including a corner, and natural (stress) boundary conditions on the rest.

Remark 4.1 *We conjecture that, for no-slip b.c., it is still possible to avoid the special class $\mathbb{P}(\Omega)$. Let $\overline{\Phi}_N \subseteq H_0^1(\Omega)$ and $\overline{\Psi}_N \subseteq H^1/\mathbb{R}(\Omega)$ and require $\partial q/\partial n = 0$ for all $q \in \overline{\Psi}_N$. Then $v \cdot n = 0$ already for any $v \in X_N$ and we can enforce $v \cdot \tau = 0$ by requiring $\phi_n = -\psi_\tau$ on Γ. We exhibit the said construction for Ω a square, extending the analysis above to the class $\overline{\mathbb{P}}(\Omega) = \{p \in Y \cap C^0(\overline{\Omega}) : p(\pm 1, \pm 1) = 0\}$.*

Proof Given a $\psi \in \overline{\Psi}_N$ (that may approximate the exact potential of the pressure optimally), our task is to construct a $\phi \in \overline{\Phi}_N$ so that there is compatibility:

$$\frac{\partial \phi}{\partial n} = -\frac{\partial \psi}{\partial \tau} \text{ on } \partial\Omega. \tag{30}$$

Towards this end allow us some notation: let ℓ_i be the ith Legendre polynomial and

$$L_i(t) = \int_{-1}^{t} \ell_{i-1}, \text{ for } i \geq 1$$

and $L_0 = \ell_0$ so that

$$\begin{aligned} L_i &= \frac{1}{2i-1}(\ell_i - \ell_{i-2}), \text{ for } i \geq 2, \\ L_1 &= \ell_0 + \ell_1, \text{ and } L_0 = \ell_0. \end{aligned}$$

Then, let us begin with the general description of ψ: this function may be represented as

$$\psi(x,y) = \sum_{i,j=0}^{p} \beta_{ij} L_i(x) L_j(y)$$

subject to the requirements that:

$$\int_\Omega \psi = \sum_{i,j=0}^{2} \beta_{ij} \left(\int_{-1}^{1} L_i\right)\left(\int_{-1}^{1} L_j\right) = 0$$

(since $\int_{-1}^{1} L_i = 0$ for $i > 2$) as well as

$$\frac{\partial \psi}{\partial n} = 0 \text{ on } \partial\Omega,$$

which can be verified to be satisfied iff, $\forall i,j$,

$$\sum_{i=1, i\,\text{odd}}^{p} \beta_{ij} = \sum_{i=1, i\,\text{even}}^{p} \beta_{ij} = \sum_{j=1, i\,\text{odd}}^{p} \beta_{ij} = \sum_{j=1, i\,\text{even}}^{p} \beta_{ij} = 0.$$

These constraints already imply that $\Delta\psi(\pm 1, \pm 1) = 0$. To such a ψ we wish to find a $\phi \in \overline{\Phi}_N$ so that $\phi_n = -\psi_\tau$ on the boundary of Ω: ϕ can generally be expressed as:

$$\phi(x,y) = \sum_{i,j=0}^{p} \alpha_{ij} L_i(x) L_j(y)$$

subject to the requirement that:

$$\phi = 0 \text{ on } \partial\Omega,$$

which can be verified to be satisfied iff

$$\alpha_{ij} = 0 \text{ if one or both of } i \text{ and } j \in \{0,1\}.$$

Hence a general ϕ takes the form:

$$\phi(x,y) = \sum_{i,j=2}^{p} \alpha_{ij} L_i(x) L_j(y).$$

We then list the identities that ϕ must satisfy resulting from requiring (30) on each of the four boundary segments: First, on $x = -1$,

$$\begin{aligned}
\phi_n &= -\phi_x \\
&= -\sum_{i,j=2}^{p} \alpha_{ij} \ell_{i-1}(-1) L_j(y) \\
&= -\sum_{j=0}^{1} \frac{1}{2j+3} \left(\sum_{i=2}^{p} (-1)^i \alpha_{ij}\right) \ell_j \\
&\quad + \sum_{j=2}^{p-2} \left(\sum_{i=2}^{p} (-1)^i \left(\frac{\alpha_{ij}}{2j-1} - \frac{\alpha_{i,j+2}}{2j+3}\right)\right) \ell_j \\
&\quad + \sum_{j=p-1}^{p} \frac{1}{2j-1} \left(\sum_{i=2}^{p} (-1)^i \alpha_{ij}\right) \ell_j
\end{aligned}$$

and the expression for ψ_τ is

$$-\psi_y = -\sum_{i,j=0}^{p} \beta_{ij} L_i(-1) \ell_{j-1}(y) = -\sum_{j=0}^{p-1} \beta_{0,j+1} \ell_j(y)$$

and we equate like terms to obtain the following final equations.

On $x = -1 : \forall j \geq 1$

$$\frac{1}{4j+1} \sum_{i=2}^{p} (-1)^i \alpha_{i,2j+1} = \sum_{k=j+1}^{[p/2]} \beta_{0,2k},$$

$$\frac{1}{4j-1} \sum_{i=2}^{p} (-1)^i \alpha_{i,2j} = \sum_{k=j}^{[(p-1)/2]} \beta_{0,2k+1},$$

with a summation over an empty set convened to be zero and no single index of α or β larger than p allowed. One makes use of the fact that $\sum_j \beta_{0,j} = 0$ to ensure that the two lowest-order sums above (resulting from taking $j = 1$) are not over-determined. Similarly,

On $x = +1 : \forall j \geq 1$

$$\frac{1}{4j+1} \sum_{i=2}^{p} \alpha_{i,2j+1} = -\sum_{k=j+1}^{[p/2]} \beta_{0,2k} + \beta_{1,2k},$$

$$\frac{1}{4j-1} \sum_{i=2}^{p} \alpha_{i,2j} = -\sum_{k=j}^{[(p-1)/2]} \beta_{0,2k+1} + \beta_{1,2k+1}.$$

To prevent the two lowest-order sums above (resulting from taking $j = 1$) from being over-determined, we now also use that $\sum_j \beta_{1,j} = 0$. Also,

On $y = -1 : \forall i \geq 1$

$$\frac{1}{4i+1} \sum_{j=2}^{p} (-1)^j \alpha_{2i+1,j} = \sum_{k=i+1}^{[p/2]} \beta_{2k,0},$$

$$\frac{1}{4i-1} \sum_{j=2}^{p} (-1)^j \alpha_{2i,j} = \sum_{k=i}^{[(p-1)/2]} \beta_{2k+1,0},$$

and, finally,

On $y = +1 : \forall i \geq 1$

$$\frac{1}{4i+1} \sum_{j=2}^{p} \alpha_{2i+1,j} = -\sum_{k=i+1}^{[p/2]} \beta_{2k,0} + \beta_{2k,1},$$

$$\frac{1}{4i-1} \sum_{j=2}^{p} \alpha_{2i,j} = -\sum_{k=i}^{[(p-1)/2]} \beta_{2k+1,0} + \beta_{2k+1,1},$$

again using that certain β-sums vanish when one index is frozen at zero. It is clear that, for $p \geq 5$, we may solve this system for α (for $p > 5$, there is more than one solution, and interestingly, for $p = 4$ the system is over-determined). Of course, we are still restricting the function values at the corners, in fact $\nabla\phi = \nabla\psi = 0$ at the four corners, and also the pressure (as $\Delta\psi$) is forced to be zero there. We may factor out this proviso with the help of the following remark. □

Remark 4.2 *Obviously, in the present situation with no-slip b.c., the corners of Ω are classically known to be singular boundary vertices (with the number of elements abutting the vertex $k = 1$), cf. [22] and [26]. Before we pass on to the natural remedy: more elements, we note that it is possible to remove the requirement that the continuous pressure (if it is smooth enough) be zero at the corners of Ω through a slight extension of the present construction.*

Proof Let the pressures be augmented by the set of bilinear functions, $\hat{Y}_N = \overline{Y}_N \oplus \mathcal{Q}^1 \cap Y$. Also let the velocity space be augmented by biquartics, $\hat{X}_N = \overline{X}_N \oplus [\mathcal{Q}^4]^2 \cap X$. ($\overline{Y}_N = \Delta\overline{\Psi}_N$ and $\overline{X}_N = \nabla \times \overline{\Phi}_N \oplus \nabla\overline{\Psi}_N$.) Then, the L^2-orthogonal decomposition: $\forall q \in \hat{Y}_N, \exists q_1 \in Y_N, q_2 \in \mathcal{Q}^1 : q = q_1 + q_2$, holds as $(q_1, q_2) = (\Delta\psi_1, q_2) = -(\nabla\psi_1, \nabla q_2) + < \partial\psi_1/\partial n, q_2 >= (\psi_1, \Delta q_2) - < \psi_1, \partial q_2/\partial n >= 0$. By results in [26], [25], [22], there holds the divergence-stability: $\forall q_2 \in \mathcal{Q}^1, \exists u_2 \in [\mathcal{Q}^4]^2 \cap X$: div $u_2 = q_2$ and $\|u_2\|_X \leq C\|q_2\|_Y$ with C independent of p. Actually, bicubics would suffice due to the stability of $([\mathcal{Q}^2]^2 \cap X, \mathcal{Q}^0 \cap Y)$. The already established divergence-stability of the pair (X_N, Y_N) yields similarly a u_1 associated with q_1, and we obtain, with $u = u_1 + u_2$, that $\operatorname{div} u = q$ and

$$\begin{aligned} \|u\|_X^2 &= \|u_1 + u_2\|_X^2 \leq 2\left(\|u_1\|_X^2 + \|u_2\|_X^2\right) \\ &\leq C\left(\|q_1\|_Y^2 + \|q_2\|_Y^2\right) = C\|q\|_Y^2, \end{aligned}$$

establishing combined divergence-stability of the pair $(\hat{X}_N, \hat{Y}_N)$. □

In this manner, it is possible to give optimal convergence rate results also for pressures not subject to corner constraints, for $P \in H^s$ with $s > 1$ directly applying the above and for $P \in H^s$ with $s \in (0, 1)$ by first modifying P near the corners. Now $\hat{X}_N \not\subseteq [H_0^1]^2$.

We note that we could just as well have used harmonic q_2's then needing to cook up corresponding velocities (why not gradients plus curls of $\Delta^{-1}q_2$?), but instead we'll go on to the more natural remedy.

4.2 Multiple elements

We refer, first, to the definitions at the beginning of section 3.2. Let Ω be a convex, polygonal domain (perhaps with piecewise curvilinear boundary Γ). We choose (for no-slip B.C.) given S^p defined in (21):

$$\text{(31)} \qquad \Phi_N = \Psi_N = S^{p+1} \cap H_0^2(\Omega), \quad \text{and then} \\ X_N = \nabla\times\Phi_N \oplus \nabla\Psi_N, \quad Y_N = \nabla\cdot X_N = \Delta\Psi_N.$$

In the second to last identity, we understand div as defined on $H(\text{div})$. Thus the discrete velocities $X_N \subseteq [C^0(\overline{\Omega})]^2$, see [7]. The discrete pressures are allowed to be discontinuous.

Remark 4.3 *We may be overshooting with C^1 elements for both Φ_N and Ψ_N – yielding C^0 ones for X_N – when it would have sufficed to have continuity of the normal components of the combined functions in X_N across inter-element boundaries. We do not know if this is possible for general elements with quasi-optimal approximation properties when we impose the additional constraint that $\phi_\tau - \psi_n$ be continuous across $\partial\Omega_i \cap \partial\Omega_j$. It is possible, however, through a similar construction as in Remark 4.1 for element divisions consisting solely of parallelograms. We can also handle the compatibility constraint (30) across the two cathetes in the standard triangle, which may suffice for many divisions.*

We state next the main result in this subsection for no-slip b.c.

Proposition 4.2 *This mixed method is well-posed and the following error estimates hold:*

$$\begin{aligned} \|u - u_N\|_X &\leq Cp^{-r}\|u\|_{r+1}, \text{ and} \\ \|P - P_N\|_Y &\leq Cp^{-r}(\|u\|_{r+1} + \|P\|_r), \end{aligned}$$

provided $P \in \mathbb{P}(\Omega)$.

Proof As in Prop. 4.1 noting that Lemma 4.1 holds as before for the new (X_N, Y_N). □

For stress-free b.c. we may redefine Φ_N and Ψ_N:

Corollary 4.2 *Let $\tilde{\Phi}_N = \tilde{\Psi}_N = S^{p+1} \cap H_0^1(\Omega)$. Then, this mixed method is well-posed and the following error estimates hold:*

$$\begin{aligned} \|u - u_N\|_X &\leq Cp^{-r}\|u\|_{r+1}, \text{ and} \\ \|P - P_N\|_Y &\leq Cp^{-r}(\|u\|_{r+1} + \|P\|_r). \end{aligned}$$

Remark 4.4 *We conjecture that, for no-slip b.c., it is still possible to avoid the special class $\mathbb{P}(\Omega)$ - and not only by the means mentioned in Remarks 4.1 and 4.2. It might* *be possible now to approximate quasi-optimally by using solutions to Poisson problems with homogeneous Cauchy data on the boundary Γ in the elements abutting Γ, which are chosen to preclude the existence of singular boundary vertices. This is merely a conjecture.*

Remark 4.5 *We have actually not taken advantage of the freedom in selecting $\Phi_N \neq \Psi_N$, allowing for some interesting possibilities (de-emphasizing pressure approximation for example).*

Elements with one curved side coinciding with Γ are, once more, proposed to be taken care of as described in [9].

Finally, we refer to [29], [28], [27], [21], and [24] among others for treatments of C^1 increasing degree finite elements.

5 Concluding remarks

A number of very interesting open questions immediately present themselves. The main one is probably how well such methods would perform in practice. In a joint project with Tad Janik of University of Alabama in Huntsville we hope to address these issues.

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