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IDEAL COMPLETIONS

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The purpose of this note is to illustrate how some lattice theoretical ideas, which have not been exploited in the context of abstract (ring) ideal theory, can be put to work. Namely, we will exploit the fact that lattices of (ring) ideals are algebraic lattices.

0. Ideal completions of join-semilattices

Let P and Q be posets, $P \subseteq Q$. Q is an extension of P if the ordering of P is the restriction to P of the ordering of Q (i.e., for $x,y \in P$, $x \leq y$ in P if and only if $x \leq y$ in Q). P is join-dense in Q if every $q \in Q$ is representable as the join (in Q) of some subset $M \subseteq P$, $q = \sup_Q M$; one can then take as M the set of all elements $p \in P$ such that $p \leq q$, $M = P \cap (q]$. An element $x \in P$ is called compact if the following condition holds true for each subset $M \subseteq P$:

(0.1) if $x \leq \sup_P M$, then $x \leq \sup_P M'$ for some finite $M' \subseteq M$.

A complete lattice L is said to be algebraic if the set of compact elements, C(L), of L is join-dense. Note

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that in any complete lattice L, C(L) is a join-subsemilattice containing the least element of L.

<u>Theorem 0.1.</u> Let P be a join-semilattice with least element o. Then there exists a complete extension I(P) of P satisfying the following conditions:

(i) P is join-dense in I(P);

(ii) the compact elements of I(P) are exactly the elements of P, $P=C(I(P))$.

Such I(P) is uniquely determined up to a unique P-isomorphism and is called "the" ideal completion of P. Note that I(P) is an algebraic lattice. As a consequence of condition (i) P is completely meet-faithful in I(P), i.e., if $p=\inf_P M$ where $p \in P$ and $M \subseteq P$, then $p=\inf_{I(P)} M$. So, in particular, if P has e as largest element, then e is also the largest element of I(P). Also, as a consequence of condition (ii) P, being finitely join-closed in I(P), is finitely join-faithful, i.e., if $p=\sup_P M$ where $p \in P$ and M is a finite subset of P, then $p=\sup_{I(P)} M$. Caution: this does not necessarily hold for infinite M. But it does allow us to write $x \vee y$ and $x \wedge y$ for $x,y \in P$ without any risk of ambiguity.

The usual proof of this theorem is by construction. $I \subseteq P$ is called an <u>ideal</u> if I is a lower end (i.e., if $y \in I$ and $x \leq y$ then $x \in I$) and I is closed under finite joins. In particular, $o \in I$. Let $I(P)=\{I \mid I$ is an ideal of $P\}$. For $p \in P$, let $(p]=\{q \mid q \in P,\ q \leq p\}$. Then $(p] \in I(P)$ and the mapping $p \longmapsto (p]$ is an embedding of P into $I(P)$. One shows $I(P)$ satisfies conditions (i) and (ii) of the theorem. For a more detailed exposition, cf. [2].

As an immediate consequence of Theorem 0.1 we obtain the following corollary:

Corollary 1. Each algebraic lattice L is the ideal completion of the semilattice $C(L)$, $L=I(C(L))$.

Henceforth we will use the term semilattice to mean join-semilattice with least element o.

The aforementioned uniqueness of the ideal completion is a special case of the following universal property:

Theorem 0.2. Let L be a complete lattice, P a subsemilattice of L containing the least element. Then the following statements are equivalent:

(i) $L=I(P)$;

(ii) for each complete lattice F, each finitely join-preserving mapping $\varphi: P \longrightarrow F$, there is exactly one completely join-preserving mapping $\psi: L \longrightarrow F$ extending φ.

Note that φ finitely join-preserving means that $\varphi(x \vee y)=\varphi(x) \vee \varphi(y)$ and $\varphi(o)=o$. The statement of this theorem verbatim can be found in Schmidt [4].

The proof of (i) $\Longrightarrow$ (ii) is again by construction;

for x in I(P) one defines $\Psi(x)=\sup_F \varphi(P\cap(x])$. Then one checks that Ψ is the unique completely join-preserving extension of φ. For the proof of (ii)$\Longrightarrow$(i) one uses the standard universal algebra device for universal solutions.

1. Ideal completions of sl-semigroups

A <u>semilattice-semigroup</u> S or, in short, an <u>sl-semigroup</u> is a (join-) semilattice and at the same time a semigroup (in multiplicative notation) subject to the following compatibility conditions:

(i) for any $x,y,z\in S$, $x(y\vee z)=xy\vee xz$, $(y\vee z)x=yx\vee zy$;

(ii) for any $x\in S$, $xo=ox=o$.

(i) and (ii) may be combined in the statement that the product xy as a function of one of its factors is finitely join-preserving. As a consequence, multiplication with an element, be it on the right or the left, is order preserving.

Let I(S) be the ideal completion of S. We would like to extend the multiplication to I(S) so that it also becomes an sl-semigroup.

Note that for $x,y\in S$, $xy=\max\{x'y' \mid x'\in(x]\cap S, y'\in(y]\cap S\}=\sup_{I(S)}\{x'y' \mid x'\in(x]\cap S, y'\in(y]\cap S\}$.

Thus, if we define

(1.1) $xy=\sup_{I(S)}\{x'y' \mid x'\in(x]\cap S,\ y'\in(y]\cap S\}$

for $x,y\in I(S)$ we indeed obtain an extension of the multiplication on S. Let $x,y,z\in I(S)$. First, we prove that multiplication by an element is order preserving. Assume $x\leq y$, let $x'\in(x]\cap S$, $z'\in(z]\cap S$. Then $x'\in(y]\cap S$, so, $x'z'\leq yz$, thus, $xz\leq yz$. Similarly, $zx\leq zy$. Next we show that for any $M\subseteq S$,

(1.2) if $y=\sup_{I(S)}M$, then $xy=\sup_{I(S)}xM$, $yx=\sup_{I(S)}Mx$.

Clearly, $xy\geq\sup_{I(S)}xM$. Conversely, let $x'\in(x]\cap S$, $y'\in(y]\cap S$. Then $y'\leq\sup_{I(S)}M$, but by compactness there exists $M'\subseteq M$, finite, such that $y'\leq\sup_{I(S)}M'=\sup_S M'$, so $x'y'\leq x'\sup_S M'=\sup_S x'M'\leq\sup_{I(S)}xM$. The proof of the other half is alike. Now we are ready to prove associativity. By (1.1) $yz=\sup_{I(S)}\{y'z' \mid y'\in(y]\cap S,\ z'\in(z]\cap S\}$, so, by (1.2), $x(yz)=\sup_{I(S)}\{x(y'z') \mid y',z' \text{ as above}\}$. But for $x'\in(x]\cap S$, $x'(y'z')=(x'y')z'\leq(xy)z$. So, $x(y'z')\leq(xy)z$, thus $x(yz)\leq(xy)z$. Similarly, $(xy)z\leq x(yz)$. Finally, since (1.2) implies that $xo=ox=o$, it is enough to show that $x(y\vee z)=xy\vee xz$ and $(y\vee z)x=yx\vee zx$. Clearly, $x(y\vee z)\geq xy\vee xz$. On the other hand, $y\vee z=\sup_{I(S)}\{y'\vee z' \mid y',z' \text{ as above}\}$. Thus, by (1.2), $x(y\vee z)=\sup_{I(S)}\{x(y'\vee z')\}$. But for $x'\in(x]\cap S$, $x'(y'\vee z')=x'y'\vee x'z'\leq xy\vee xz$. Therefore, we have that I(S) with the multiplication defined by (1.1) is an sl-semigroup. Yet we are ready to prove a stronger result than (i). By a <u>strong</u> sl-semigroup we mean a complete sl-semigroup (completeness refers here to the semilattice structure), where multiplication by an element is completely join-preserving. We are now going to show that I(S) is a strong sl-semigroup:

(1.3) $(\bigvee_{\alpha} y_{\alpha})x = \bigvee_{\alpha} y_{\alpha}x$, and $x(\bigvee_{\alpha} y_{\alpha}) = \bigvee_{\alpha} xy_{\alpha}$,

for $x, y_{\alpha} \in I(S)$. Let $y' \leq \bigvee_{\alpha} y_{\alpha}$ and $y' \in S$. Then, by compactness, there exist $y_{\alpha_1}, \ldots, y_{\alpha_n}$ such that $y' \leq y_{\alpha_1} \vee \ldots \vee y_{\alpha_n}$ so, $xy' \leq x(y_{\alpha_1} \vee \ldots \vee y_{\alpha_n}) = xy_{\alpha_1} \vee \ldots \vee xy_{\alpha_n} \leq \bigvee_{\alpha} xy_{\alpha}$.

Suppose we have defined a multiplication, say *, on I(S) such that it extends the multiplication on S and makes I(S) into a strong sl-semigroup; since $x = \sup_{I(S)}(x] \cap S$ and $y = \sup_{I(S)}(y] \cap S$, $x * y = \sup_{I(S)}\{x' * y \mid x' \in (x] \cap S\} = \sup_{I(S)}\{x'y' \mid x' \in (x] \cap S,\ y' \in (y] \cap S\} = xy$. Thus, the following theorem is now clear:

<u>Theorem 1.1.</u> Let S be an sl-semigroup. Then there is exactly one way of extending the multiplication to I(S) so that I(S) becomes a strong sl-semigroup.

The reader may note that the proof of Theorem 1.1 is similar to proving Theorem 0.2 for mappings of two variables. Actually, an alternate proof may be based on Theorem 0.2. However, this would be no shorter than the given one.

Note that I(S) is a commutative semigroup if and only if S is. Also, if S is a monoid, then its identity 1 is also the identity of I(S). Note that 1 need not be the largest element.

Putting Theorem 1.1 together with Corollary 1 of Theorem 0.1 we get:

Corollary 1. Let L be a strong sl-semigroup which is an algebraic lattice. Assume that C(L) is a subsemigroup. Then $L=I(C(L))$.

The equality above is meant not only as lattices, but as sl-semigroups.

We also obtain the following result corresponding to Theorem 0.2:

Theorem 1.2. Let L be a strong sl-semigroup, S an sl-subsemigroup of L. Then the following statements are equivalent:

(i) $L=I(S)$;

(ii) for each strong sl-semigroup F, and each sl-homomorphism $\varphi:S\longrightarrow F$, there is exactly one strong sl-homomorphism $\psi:L\longrightarrow F$ extending φ.

By an sl-homomorphism we mean, of course, a semigroup homomorphism that is finitely join-preserving. If it is completely join-preserving we call it strong. For the proof of (i)$\Longrightarrow$(ii) it is enough to show that the ψ

given by Theorem 0.2 is a semigroup homomorphism. Let $x,y \in I(S)$. Then $\Psi(xy)=\Psi(\sup_{I(S)}(((x] \cap S)((y] \cap S))=\sup_F \Psi(((x] \cap S)((y] \cap S))=\sup_F(\Psi((x] \cap S)\,\Psi((y] \cap S))=(\sup_F \Psi((x] \cap S))(\sup_F \Psi((y] \cap S))=\Psi(x)\Psi(y)$. The proof of (ii)$\Longrightarrow$(i) is, again, by the device for universal solutions.

Let us close with some examples.

First, let us consider an arbitrary complete lattice L, and a pre-fixed non-compact element c L. We make L a strong sl-semigroup by the following multiplication: xy=c when neither x nor y is o and xy=o otherwise. This shows that in a given strong sl-semigroup, the compact elements need not always be a subsemigroup, even if it is algebraic.

Next, let us consider a complete lattice L. Let L* be the set of completely join-preserving mappings of L into itself. L*, then, is, as a subset of the complete lattice L^L, at least a poset. Being closed under arbitrary joins in L^L, L* is actually a complete lattice itself. Composition makes it a strong sl-monoid. The Cayley representation can be used to show that any strong sl-monoid L is embeddable in L*.

Let us now consider a commutative ring R with identity 1. Let K be a unitary (associative) algebra over R. $L(K)$ will denote the lattice of R-submodules of K. $L(K)$ is an algebraic lattice where $C(L(K))$ (which we will write $C(K)$ for short) is the set of finitely generated submodules. For $M,N \in L(K)$ let MN be the submodule generated by the set of all mn where $m \in M$ and $n \in N$. This multiplication makes $L(K)$ into a strong sl-monoid (with identity $R1_K$), where, moreover, $C(K)$ is an sl-submonoid. Thus, by Corollary 1 of Theorem 1.1, $L(K)=I(C(K))$. This was actually the kind of example that led to the present formal considerations.

Finally, let D be an integral domain, and K its field of quotients. So K is an algebra over D. D is a Prüfer domain (cf. [1]) if and only if $C^*(K)$ ($=C(K)\setminus\{o\}$) is a group. But then $C^*(K)$ is a lattice-ordered group (l-group). Thus,

<u>Theorem 1.3.</u> Let D be an integral domain with field of quotients K. Then D is Prüfer if and only if $L(K)=I(G)$ for some Abelian l-group G with o.

By an l-group with o we mean, of course, an l-group with an element o added to it acting both as a zero for the semigroup and the semilattice structures.

By a theorem of Jaffard (cf. [1]), for every Abelian l-group G with o, there exists a Bezout domain D with field of quotients K such that $L(K)=I(G)$ (or, equivalently C(K)=G, or L(D)=I(G-), where G- denotes the negative cone of G). Thus, from the sl-monoid point of view, there is absolutely no difference between Prüfer and Bezout domains. Similarly, there is no difference between Dedekind domains and principal ideal domains. In particular, one cannot detect principal submodules in L(K) (cf. [5]).

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